Math 407 – Linear Optimization

Today: The Simplex Method
(starting from a feasible basis)

LP: $\max c^T x$ s.t. $Ax=b$, $x \geq 0$

PIVOT STEP

Input: a feasible basis B with basic feas. sol. $v \in \mathbb{R}^n$, and a choice of "pivot rule"

Output: either

- a certificate that v is the optimal solution, or
- a nonzero vector $w \in \mathbb{R}^n$ s.t. $c^T w > 0$ and $v + \lambda w$ feas. for all $\lambda \geq 0$, or

Note this implies that the LP is unbounded! $c^T(v+\lambda w) = c^T v + \lambda c^T w \to \infty$ as $\lambda \to \infty$

- a feasible basis $\tilde{B} \neq B$ with basic feas. sol. $\tilde{v} \in \mathbb{R}^n$ and $c^T v \geq c^T \tilde{v}$

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(1) Compute the simplex tableau $T(B)$

$$T(B): \quad \begin{aligned} x_B &= p + Q x_N \\ z &= z_0 + r^T x_N \end{aligned}$$

(2) If $r \leq 0$, $c^T x = z_0 + r^T x_N \leq z_0$ on feas. region

$\Rightarrow v$ optimal (STOP)

(3) Choose $i \in N$ with $r_i > 0$ (x_i = "entering variable")

(4) If all entries of $q_i = i^{th}$ col of Q are ≥ 0, then

$x_B = p + \lambda q_i \geq 0$ for all $\lambda \geq 0 \Rightarrow$ LP unbounded (STOP)

($v + \lambda w$ feas. $\forall \lambda \geq 0$ where $w_B = q_i$, $w_i = 1$, $w_{N \setminus i} = 0$)

(5) Find $\lambda^* = \max\{\lambda : p + \lambda q_i \geq 0\}$ $\left(= \min\{p_k / |q_{ik}| : q_{ik} < 0\}\right)$

(6) <u>Choose</u> $j \in B$ s.t. j^{th} coord of $p + \lambda^* q_i$ is zero
(x_j = "leaving variable")

(7) Output $\tilde{B} = (B \cup \{i\}) \setminus \{j\}$

Remarks:

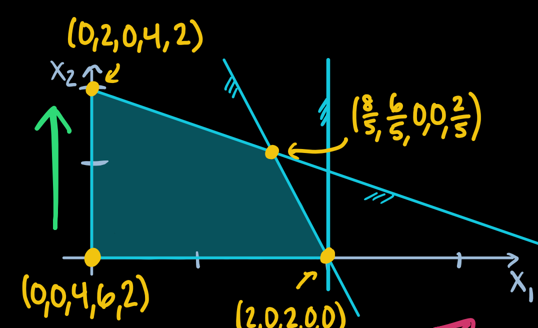
- $r_i > 0 \Rightarrow$ increasing x_i while keeping all other nonbasic var $= 0$ increases the objective function
- $x_j \geq 0$ imposes nontrivial constraint on $x_B = p + \lambda q_i$ and $p + \lambda^* q_i \geq 0 \Rightarrow \tilde{B}$ is a feasible basis
(See Lemma 5.6.1 for more detail)
- "Pivot rule" = rule for chosing i, j when there is more than one valid choice. There are many options, each w/advantages & disadvantages
(See §5.7 for more)

Ex: max $x_1 + x_2$ s.t. $Ax = b$, $x \geq 0$ where

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix}$$

$B = \{3,4,5\}$ $N = \{1,2\}$

$T(B)$:
$$x_3 = 4 - x_1 - 2x_2$$
$$x_4 = 6 - 3x_1 - x_2$$
$$x_5 = 2 - x_1$$
$$z = \underline{x_1} + \underline{x_2}$$



could choose x_1 or x_2 to enter

c

Choose x_2 to enter: (keep $x_1=0$)

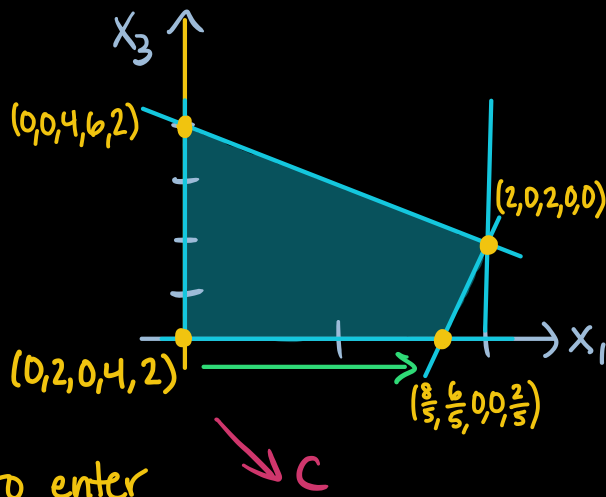
$$x_3 = 4 - 2x_2 \to x_2 \le 2 \; (*)$$
$$x_4 = 6 - x_2 \to x_2 \le 6$$
$$x_5 = 2$$

max feasible val of x_2 is 2
$\Rightarrow$ makes $x_3=0$
$\Rightarrow$ x_3 leaves

$B=\{2,4,5\}$ $N=\{1,3\}$

$T(B)$:
$$x_2 = 2 - \tfrac{1}{2}x_1 - \tfrac{1}{2}x_3$$
$$x_4 = 4 - \tfrac{5}{2}x_1 + \tfrac{1}{2}x_3$$
$$x_5 = 2 - x_1$$
$$z = 2 + \underline{\tfrac{1}{2}}x_1 - \tfrac{1}{2}x_3$$

positive $\Rightarrow$ choose x_1 to enter



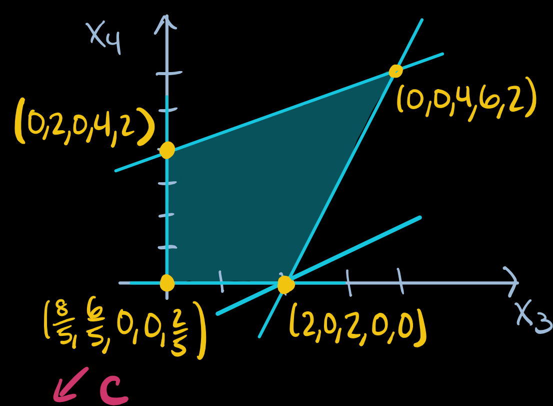
Choose x_1 to enter (keep $x_3=0$)

$$x_2 = 2 - \tfrac{1}{2}x_1 \to x_1 \le 4$$
$$x_4 = 4 - \tfrac{5}{2}x_1 \to x_1 \le \tfrac{8}{5} \; (*)$$
$$x_5 = 2 - x_1 \to x_1 \le 2$$

max feasible val of x_1 is $8/5$
$\Rightarrow$ makes $x_4=0$
$\Rightarrow$ x_4 leaves

$B=\{1,2,5\}$ $N=\{3,4\}$

$T(B)$:
$$x_1 = 8/5 + 1/5\, x_3 - 2/5\, x_4$$
$$x_2 = 6/5 - 3/5\, x_3 + 1/5\, x_4$$
$$x_5 = 2/5 - 1/5\, x_3 + 2/5\, x_4$$
$$z = \boxed{14/5} \underline{- 2/5\, x_3 - 1/5\, x_4}$$



All coeff are ≤ 0 $\Rightarrow$ basic feas. sol $(\frac{8}{5},\frac{6}{5},0,0,\frac{2}{5})$ is optimal! with opt. value $14/5$

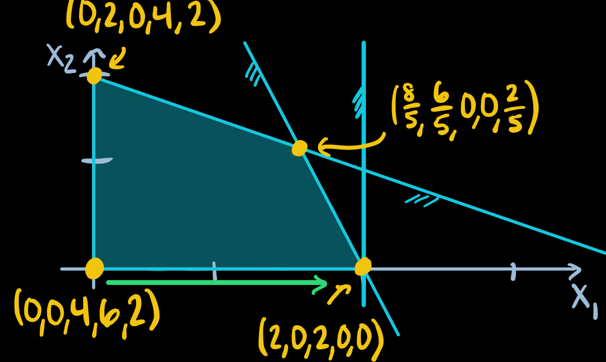
Questions:

- How do we find a basis to start with?
- Is this process guaranteed to finish?
- If so, does it finish quickly?

What if we had increased x_1 in the first pivot in the example above?

$B=\{3,4,5\}$ $N=\{1,2\}$

$T(B)$:
$$\begin{aligned} x_3 &= 4 - x_1 - 2x_2 \\ x_4 &= 6 - 3x_1 - x_2 \\ x_5 &= 2 - x_1 \\ z &= \underline{\underline{x_1}} + x_2 \end{aligned}$$



Increase x_1 (keep $x_2=0$)

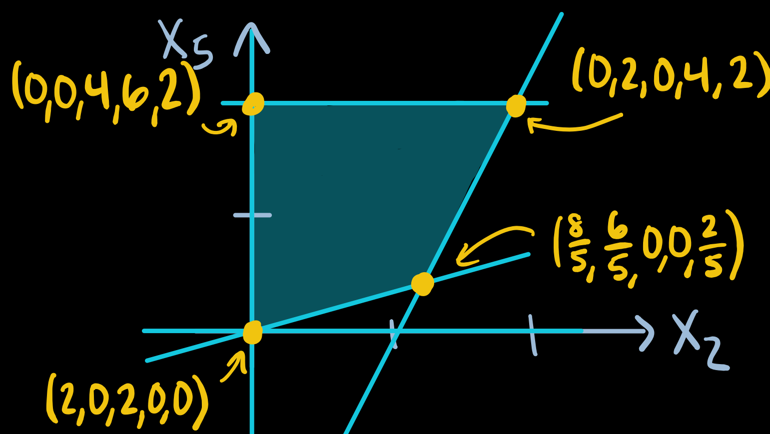
$$\begin{aligned} x_3 &= 4 - x_1 \rightarrow x_1 \le 4 \\ x_4 &= 6 - 3x_1 \rightarrow x_1 \le 2 \\ x_5 &= 2 - x_1 \rightarrow x_1 \le 2 \end{aligned}$$

max feas. val of $x_2 = 2$
$\Rightarrow$ makes both $x_4=0$, $x_5=0$
$\Rightarrow$ can choose either to leave

Choose x_5 to leave

$B=\{1,3,4\}$ $N=\{2,5\}$

$T(B)$:
$$\begin{aligned} x_1 &= 2 - x_5 \\ x_3 &= 2 - 2x_2 + x_5 \\ x_4 &= -x_2 + 3x_5 \\ z &= 2 + \underline{\underline{x_2}} - x_5 \end{aligned}$$



Choose x_2 to enter (keep $x_5=0$)

$$\begin{aligned} x_1 &= 2 \\ x_3 &= 2 - 2x_2 \Rightarrow x_2 \le 1 \\ x_4 &= -x_2 \Rightarrow x_2 \le 0 \end{aligned}$$

max feas. val of $\underline{\underline{x_2 = 0}}$
$\Rightarrow$ makes $x_4 = 0$
$\Rightarrow$ x_4 leaves

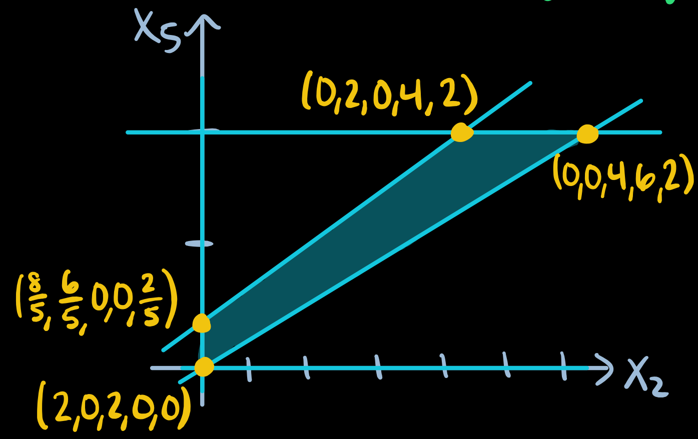
basic feas. sol.

doesn't change

$B=\{1,2,3\}$ $N=\{4,5\}$

$T(B)$

$$x_1 = 2 - x_5$$
$$x_2 = -x_4 + 3x_5$$
$$x_3 = 2 + x_4 - 5x_5$$
$$z = 2 - x_4 + \underline{2x_5}$$



Choose x_5 to enter (keep $x_4=0$)

$x_1 = 2 - x_5 \rightarrow x_5 \le 2$

$x_2 = 3x_5 \rightarrow x_5 \ge 0$

$x_3 = 2 - 5x_5 \rightarrow x_5 \le \frac{2}{5}$

max feas val of $x_5 = 2/5$

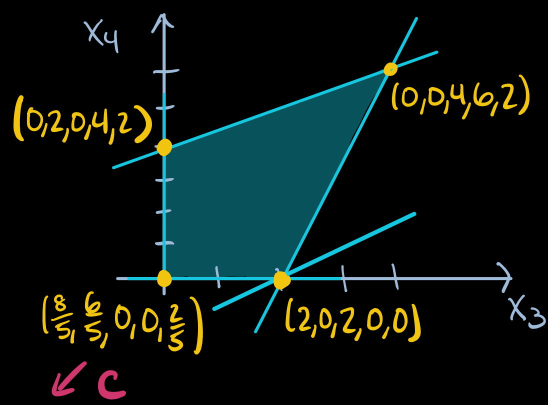
$\Rightarrow$ make $x_3 = 0$

$\Rightarrow$ x_3 leaves

$B=\{1,2,5\}$ $N=\{3,4\}$

$T(B)$:

$$x_1 = 2 - x_5$$
$$x_2 = 6/5 - 3/5\,x_3 + 1/5\,x_4$$
$$x_5 = 2/5 - 1/5\,x_3 + 2/5\,x_4$$
$$z = \boxed{14/5} \underline{- 2/5\,x_3 - 1/5\,x_4}$$



All coeff are ≤ 0 $\Rightarrow$ basic feas. sol $(\frac{8}{5},\frac{6}{5},0,0,\frac{2}{5})$ is optimal! with opt. value $14/5$

Note: The basic feasible solution $v=(2,0,2,0,0)$ corresponds to three different feasible bases: $\{1,2,3\}$, $\{1,3,4\}$, $\{1,3,5\}$