Math 407 – Linear Optimization

LP: $\max c^T x$ s.t. $Ax = b$, $x \geq 0$

where $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $\operatorname{rank}(A) = m$

Recall: $B \subseteq \{1, \dots, n\}$ with $|B|$ is a <u>basis</u> if A_B is invertible

$\Leftrightarrow$ unique $v \in \mathbb{R}^n$ with $Av = b$ with $v_j = 0$ for $j \notin B$ $\quad v_B = A_B^{-1} b$

B is a <u>feasible basis</u> if $v \geq 0$

$\Leftrightarrow$ v is a <u>basic feasible solution</u> of the LP

$x_B = (x_j : j \in B)$ "basic variables" $\quad |B| = m$

$x_N = (x_j : j \in N)$ "non-basic variables" $\quad N = \{1, \dots, n\} \setminus B$

$|N| = n - m$

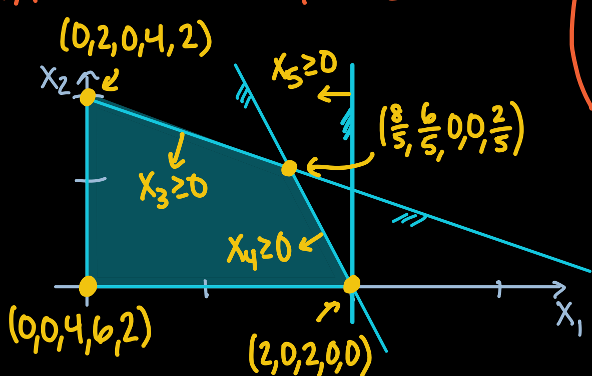
Non-basic variables as coordinates

We can parametrize $\{x \in \mathbb{R}^n : Ax = b\}$ with non-basic var x_N:

$$Ax = A_B x_B + A_N x_N = b \iff x_B + A_B^{-1} A_N x_N = A_B^{-1} b$$

$$\iff x_B = A_B^{-1} b - A_B^{-1} A_N x_N$$

Ex: $\max x_1 - x_2$ s.t. $\begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix}$ $\quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$



$B = \{3, 4, 5\}$

$x_3 = 4 - x_1 - 2x_2$

$x_4 = 6 - 3x_1 - x_2$

$x_5 = 2 - x_1$

basic feas. sol. given by B: $(0, 0, 4, 6, 2)$ $\quad$ non basic var $x_1 = 0$, $x_2 = 0$

Def: The *simplex tableau*, $T(B)$, of a feasible basis $B \subseteq \{1,\dots,n\}$ is a system of $m+1$ equations in the $n+1$ variables $x_1,\dots,x_n, z$ with

1) the same set of solutions as $Ax=b$, $z=c^Tx$
2) the form
$$x_B = p + Qx_N$$
$$z = z_0 + r^T x_N$$

with $p \in \mathbb{R}^m$, $r \in \mathbb{R}^{n-m}$, $Q \in \mathbb{R}^{m\times(n-m)}$, $z_0 \in \mathbb{R}$

Idea: $Ax=b \iff x_B = \underbrace{A_B^{-1}b}_{p} - \underbrace{A_B^{-1}A_N}_{Q} x_N$

$$c^Tx = c_B^T x_B + c_N^T x_N = c_B^T(A_B^{-1}b - A_B^{-1}A_N x_N) + c_N^T x_N$$
$$\Longrightarrow z_0 = c_B^T A_B^{-1} b \qquad r = c_N - (c_B^T A_B^{-1} A_N)^T$$

Basic feas. sol. corresponding to B: $(x_B, x_N) = (p, 0)$

Value of c^Tx at this point: z_0 ← why? $z = c^Tx = z_0 + r^Tx_N$
$= z_0$ when $x_N = 0$

In Ex: $T(\{3,4,5\})$

$$\begin{aligned} x_3 &= 4 - x_1 - 2x_2 \\ x_4 &= 6 - 3x_1 - x_2 \\ x_5 &= 2 - x_1 \\ z &= 0 + x_1 - x_2 \end{aligned}$$

$$x_B = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} \quad p = \begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix} \quad Q = \begin{pmatrix} -1 & -2 \\ -3 & -1 \\ -1 & 0 \end{pmatrix} \quad x_N = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$z_0 = 0 \qquad r = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$T(\{3,4,5\})$

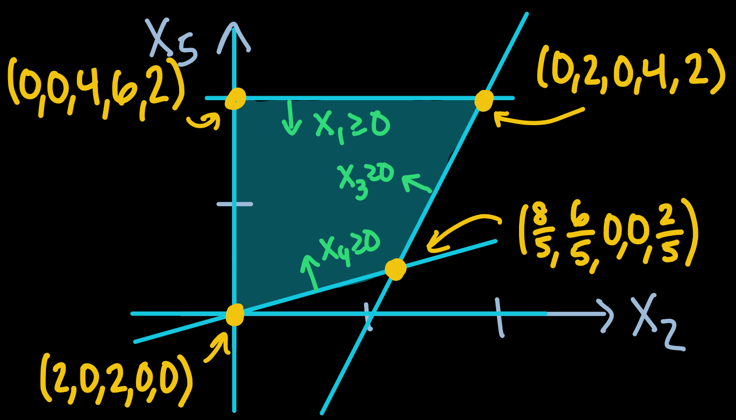
$$x_3 = 4 - x_1 - 2x_2$$
$$x_4 = 6 - 3x_1 - x_2$$
$$x_5 = 2 - x_1$$
$$z = 0 + x_1 - x_2$$

$x_1 \to 2 - x_5$

$T(\{1,3,4\})$ (use x_2, x_5 as parameters)

$$x_1 = 2 - x_5$$
$$x_3 = 4 - (2 - x_5) - 2x_2$$
$$x_4 = 6 - 3(2 - x_5) - x_2$$
$$z = 0 + (2 - x_5) - x_2$$

$$x_1 = 2 \quad - x_5$$
$$x_3 = 2 - 2x_2 + x_5$$
$$x_4 = 0 - x_2 + 3x_5$$
$$z = 2 - x_2 - x_5$$



Corresp. basic feas. sol: $(2,0,2,0,0)$ $x_2 = 0$, $x_5 = 0$

val. of $c^T x$ at ↑ : 2 (plug $x_2 = 0, x_5 = 0$ into z)

Obj function: $2 - x_2 - x_5$

Since $x_2 \geq 0$, $x_5 \geq 0$, this is ≤ 2

$\Rightarrow$ basic feas. sol. $(2,0,2,0,0)$ is optimal

More generally, if B is a feasible basis and $r \leq 0$ in $T(B)$, then

z_0 = opt. value

$(x_B, x_N) = (p, 0)$ an opt. solution

$$\left(T(B): \begin{array}{l} x_B = p + Q x_N \\ z = z_0 + r^T x_N \end{array} \right)$$

Why? obj function takes value z_0 at $(x_B, x_N) = (p, 0)$

$\Rightarrow$ opt. value $\geq z_0$

$x_N \geq 0$ on feas. region $\Rightarrow r^T x_N \leq 0$

$\Rightarrow c^T x = z_0 + r^T x_N \leq z_0$

$\Rightarrow$ opt. value $\leq z_0$

Subtlety: Different bases can give the same basic feasible solution! (when $> n-m$ coordinates are zero)

In Ex: $\{1,3,4\} \xrightarrow{x_2=x_5=0} v=(2,0,2,0,0)$

$\{1,3,5\} \xrightarrow{x_2=x_4=0} v$

$\{1,2,3\} \xrightarrow{x_4=x_5=0} v$

These all correspond to the same basic feas. sol. v but their tableaux will look different!

$T(\{1,2,3\})$

$$
\begin{aligned}
x_1 &= 2 \qquad\qquad - x_5 \\
x_2 &= 0 - x_4 + 3x_5 \\
x_3 &= 2 + 2x_4 - 5x_5 \\
z &= 2 + x_4 - 4x_5
\end{aligned}
$$

basic feas. solution $v=(2,0,2,0,0)$ is optimal, but we can't see it from $T(\{1,2,3\})$!