Math 407 – Linear Optimization

Today: Vertices in eq. form & basic feasible sol'n.

Last time: We can reformulate any linear program as $\max c^T x$ s.t. $Ax=b$, $x \geq 0$.

Preprocess (w/ Gaussian elimination) to check $\{x \in \mathbb{R}^n : Ax=b\}$ is nonempty and eliminate redundant equations s.t. rows of $A \in \mathbb{R}^{m \times n}$ are linearly independent.

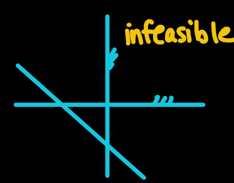
Note 1: If $Ax=b$ has no solution $x \in \mathbb{R}^n$, then the LP is infeasible. (Can check using Gaussian elimination)

$\Rightarrow$ can always assume $\{x \in \mathbb{R}^n : Ax=b\}$ nonempty

This is not enough to guarantee that LP is feasible!

Ex: $\max x_1$ s.t.

$x_1 + x_2 = -1$, $x_1 \geq 0$, $x_2 \geq 0$



Note 2: we can always assume rows of A are linearly independent (i.e. $\operatorname{rank}(A) = m$) by discarding redundant equality constraints.

Ex: $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \iff \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

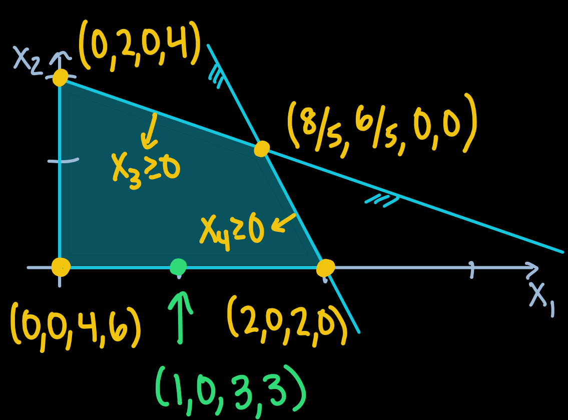
$2x_1 + x_2 + x_3 = 2$ redundant $\left(\text{implied by } \begin{matrix} x_1 + x_2 = 1 \\ x_1 + x_3 = 1 \end{matrix} \right)$

A point $v \in P = \{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\}$ is called a <u>basic feasible solution</u> of P if the columns of A_K are linear independent where $K = \{j \in \{1,..,n\} \text{ s.t. } v_j > 0\}$ and A_K is the $m \times |K|$ submatrix of A with columns indexed by K.

Equivalent def: v is a <u>basic feasible solution</u> if there is some $B \subseteq \{1,\dots,n\}$ of size $|B| = m$ s.t. $v_j = 0$ for all $j \notin B$ and the matrix $\underbrace{A_B}_{\text{size } m\times m}$ is invertible.

Why is this equivalent?

Ex: max $x_1 - x_2$ s.t. $x_1 \geq 0,\ x_2 \geq 0,\ x_1 + 2x_2 \leq 4,\ 3x_1 + x_2 \leq 6$



$$x_1 + 2x_2 + x_3 = 4,\quad 3x_1 + x_2 + x_4 = 6$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0$$

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$v = (2,0,2,0) \rightarrow K = \{1,3\} \rightarrow A_K = \begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix}$ ← lin. indep col $\Rightarrow$ basic feas. sol.

$v = (1,0,3,3) \rightarrow K = \{1,3,4\}$ $A_K = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$ ← col's not linearly indep $\Rightarrow$ not a basic feas. sol.

<u>Thm</u>: For $P = \{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\}$, a point $v \in P$ is a basic feasible solution of P if and only if v is a vertex of P.

(Idea of proof) For $v \in P$, $K = \{j \text{ s.t. } v_j > 0\}$

The inequalities $x_j \geq 0$ for $j \notin K$, $Ax \leq b$, $Ax \geq b$ hold with equality at v.

So v is a vertex of P

$$\iff \text{rows of } \begin{array}{c c} & \begin{array}{cc} \{1,\dots,n\}\setminus K & K \end{array} \\ \begin{array}{c} n-|K| \\ m \end{array} & \left(\begin{array}{c|c} I & 0 \\ \hline \multicolumn{2}{c}{A} \end{array}\right) \end{array} = \begin{array}{c} n-|K| \\ m \end{array} \left(\begin{array}{c|c} I & 0 \\ \hline A_{K^c} & A_K \end{array}\right) \text{ span } \mathbb{R}^n$$

$\iff$ rows of A_K span $\mathbb{R}^{|K|}$ $\Rightarrow m \geq |K|$

$\iff \operatorname{rank}(A_K) = |K|$ $(= \dim(\text{col span}(A_K)))$

$\iff$ the $|K|$ columns of A_K are linearly indep.

We call $B \subseteq \{1, \dots, n\}$ with $|B| = m$ a <u>basis</u> if the $m \times m$ matrix A_B is nonsingular.

($\iff$ col of A_B are a basis for colspan(A).)

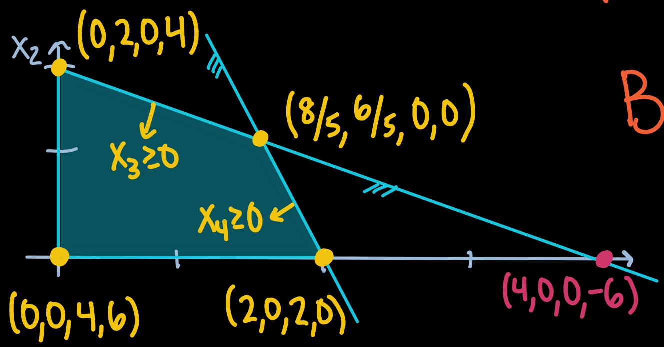
$\iff A_B x_B = b$ has unique solution $x_B \in \mathbb{R}^{|B|}$

$\iff Ax = b$ has unique solution $x \in \mathbb{R}^n$ with $x_j = 0 \ \forall j \notin B$.

B is a <u>feasible basis</u> if unique sol $x_B \in \mathbb{R}^{|B|}_{\geq 0}$.

$\Rightarrow$ unique sol'n found is a basic feasible sol'n.

Ex: $P = \{x \in \mathbb{R}^4 \text{ s.t. } \begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 6 \end{pmatrix}, \ x \geq 0\}$



$B = \{1,4\} \rightarrow A_B = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ ← invertible

$\Rightarrow$ B is a basis

$A_B x_B = b \Rightarrow \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \rightarrow$

unique sol: $\begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ \underline{\underline{-6}} \end{pmatrix}$ → B is not a feasible basis

$B = \{1,3\}$, $A_B x_B = b \leftrightarrow \begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$ unqiue sol: $\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 2 \end{pmatrix}}_{\geq 0}$

$\Rightarrow$ B is a feasible basis