Math 407 – Linear Optimization

Today: finishing up from last week and reformulating LP's

Last time: how to find vertices of a polyhedron

If P is bounded, max attained by a vertex.

One method for finding $\max c^T x$ s.t. $x \in P$ (for bounded P):

Compute all vertices of P and $c^T x$ for each, pick largest

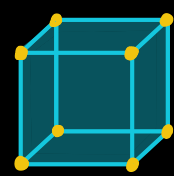
Problems? 1) only works for bounded P

2) there might be a lot of vertices

Ex: $\{x \in \mathbb{R}^n : 0 \le x_1 \le 1, \ldots, 0 \le x_n \le 1\}$

exponential in n

$\#\text{var} = n$ $\#\text{ineq} = 2n$ $\#\text{vertices} = 2^n$ even with only $2n$ ineq. constraints



Solution to Problem 2: walk between vertices rather than checking all of them

(next week!)

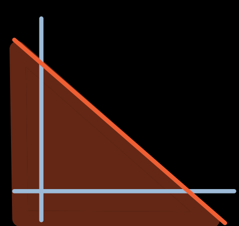
Solution to Problem 1: reformulate LP as

$$\max c^T x \text{ s.t. } Ax = b,\ x \ge 0$$

<u>Thm</u>: If the linear program $\max c^T x$ s.t. $Ax = b$, $x \ge 0$ is bounded and feasible, then the maximum is acheived by a vertex.

Compare to

$$\max x_1 + x_2 \text{ s.t. } x_1 + x_2 \le 1$$



max acheived, but not by a vertex!

(Main idea of proof): $\{x\in\mathbb{R}^n : x\geq 0\}$ doesn't contain any lines $\{v+\lambda w : \lambda\in\mathbb{R}\}$

Starting from a non-vertex v, $\exists$ nonzero $w\in\mathbb{R}^n$ s.t. $v\pm w$ feasible. Can assume $c^T w\geq 0$ (else $w\leftrightarrow -w$).

($c^T w>0$) Either $v+\lambda w$ feasible $\forall\lambda\geq 0$
($\Rightarrow$ LP unbounded, $c^T(v+\lambda w)=c^T v+\lambda c^T w\to\infty$ as $\lambda\to\infty$)
or not ($\Rightarrow$ boundary pt $v+\lambda^* w$ satisfies more "$\leq$" with "$=$" and $c^T(v+\lambda^* w)>c^T v$). Repeat starting from $v+\lambda^* w$

($c^T w=0$) $v+\lambda w$ not feasible $\forall\lambda\in\mathbb{R}$
boundary pt $v+\lambda^* w$ satisfies more "$\leq$" with "$=$" and $c^T(v+\lambda^* w)\geq c^T v$).

Reformulating LP's

Input: an LP $\max\{c^T x : Ax\leq b\}$

Output: an equivalent LP $\max\{\tilde{c}^T\tilde{x} : \tilde{A}\tilde{x}=\tilde{b}, \tilde{x}\geq 0\}$

same obj. value, easy translation between opt. solutions

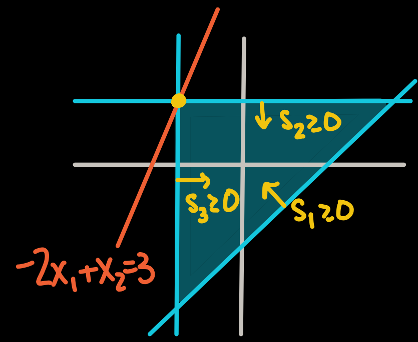
Step 1: add "slack" variables to change linear inequalities into linear equations

$$a_i^T x\leq b_i \iff \exists\, s_i\in\mathbb{R}_{\geq 0} \text{ s.t. } a_i^T x+s_i=b_i.$$

Step 2: For any variable x_i not constrained to be nonnegative, replace x_i with y_i-z_i where $y_i\geq 0$, $z_i\geq 0$.

Ex: max $-2x_1+x_2$ s.t.

$x_1-x_2\le 1,\ x_2\le 1,\ -x_1\le 1$



STEP 1

max $-2x_1+x_2$ s.t.

$x_1-x_2+s_1=1,\ x_2+s_2=1,\ -x_1+s_3=1,\ s_1\ge 0, s_2\ge 0, s_3\ge 0$

STEP 2 $x_1\to y_1-z_1$ $x_2\to y_2-z_2$

max $-2(y_1-z_1)+(y_2-z_2)$ s.t. $y_1\ge 0, z_1\ge 0, y_2\ge 0, z_2\ge 0, s_1\ge 0, s_2\ge 0, s_3\ge 0,$

$(y_1-z_1)-(y_2-z_2)+s_1=1,\ y_2-z_2+s_2=1,\ -y_1+z_1+s_3=1$

$$= \max\ (-2,2,1,-1,0,0,0)(y_1,z_1,y_2,z_2,s_1,s_2,s_3)^T \text{ s.t.}$$

$$\begin{pmatrix} 1 & -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ z_1 \\ y_2 \\ z_2 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\quad \begin{pmatrix} y_1 \\ z_1 \\ y_2 \\ z_2 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} \ge \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Opt value $=3$

Opt solution: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \longleftrightarrow \begin{pmatrix} y_1 \\ z_1 \\ y_2 \\ z_2 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}$

$\} \ y_1-z_1=x_1$

$\} \ y_2-z_2=x_2$

$\to s_1=1-x_1+x_2$

$\to s_2=1-x_2$

$\to s_3=1+x_1$

Starting from LP with n variables, m inequalities this give LP with (at most) $\underline{2n+m}$ variables and $\underline{m}$ equations.

linear in n,m