Math 407 – Linear Optimization

Today: Vertices of Polyhedra

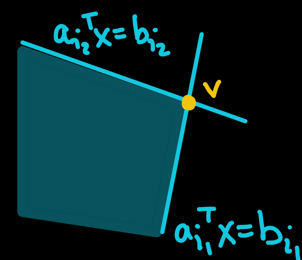
Hwk 2 due Oct 12

Vertices of Polyhedra

Given a polyhedron $P=\{x\in\mathbb{R}^n : a_1^Tx\le b_1,\dots,a_m^Tx\le b_m\}$, and a point $v\in\mathbb{R}^n$, when is v a vertex of P?

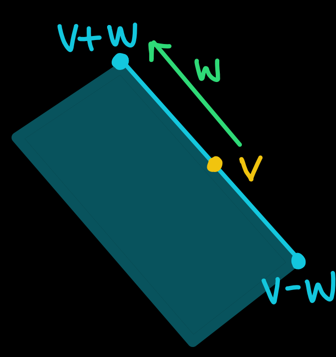
Three answers:

(1) v is a vertex of P if it satisfies all the inequalities ($a_i^Tv\le b_i$ for all i) and there are n linearly independent inequalities that hold with equality at v ($a_{i_1}^Tv=b_{i_1},\dots,a_{i_n}^Tv=b_{i_n}$ where $a_{i_1},\dots,a_{i_n}$ are linearly independent)

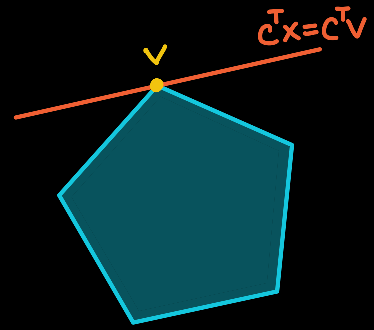


Note: If $a_{i_1},\dots,a_{i_n}$ are linearly independent then $x=v$ is the unique solution to $a_{i_1}^Tx=b_{i_1},\dots,a_{i_n}^Tx=b_{i_n}$

(2) v is a vertex of P if $v\in P$ and there is no non-zero vector w such that both $v+w$ and $v-w$ belong to P.



(3) v is a vertex of P if $v \in P$ and there is some $c \in \mathbb{R}^n$ s.t. v is the unique maximizer of $c^T x$ over $x \in P$.
That is, $c^T v > c^T u$ for all $u \in P \setminus \{v\}$



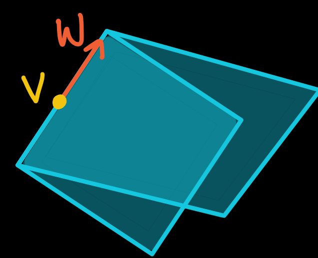
Thm: (1), (2), (3) are all equivalent.
(We could take any of these as our definition of vertex!)

(2) $\Rightarrow$ (1) (same as "not (1) $\Rightarrow$ not (2)")

Let $v \in P$ and $I = \{ i \in \{1,..,m\} \text{ s.t. } a_i^T v = b_i \}$
Suppose $\text{span}\{a_i : i \in I\}$ has $\dim < n$
$\Rightarrow \exists$ nonzero $w \in \mathbb{R}^n$ s.t. $a_i^T w = 0$ for all $i \in I$

Claim: For small enough $\varepsilon > 0$, $v \pm \varepsilon w \in P$.

For $i \in I$, $a_i^T(v \pm \varepsilon w) = a_i^T v \pm \varepsilon \underbrace{a_i^T w}_{=0} = b_i$

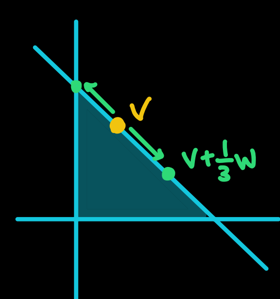


For $j \notin I$, $a_j^T v < b_j$

$\Rightarrow a_j^T(v \pm \varepsilon w) = \underbrace{a_j^T v}_{<b_j} \pm \varepsilon a_j^T w \le b_j$ for small enough ε

$< b_j$ $\rightarrow$ gives wiggle room $b_j - a_j^T v > 0$

Ex: $P = \{ x \in \mathbb{R}^2 : \underset{(1)}{x_1 \ge 0}, \underset{(2)}{x_2 \ge 0}, \underset{(3)}{x_1 + x_2 \le 1} \}$



$v = (1/3, 2/3)$ Inequalities at equality at v? (3)

$\Rightarrow \exists\, w \in \mathbb{R}^2 \setminus \{0\}$ s.t. $0 = a_3^T w = x_1 + x_2$, e.g. $w = (1,-1)$

$v + \lambda w = (1/3 + \lambda, 2/3 - \lambda) \in P \iff 1/3 + \lambda \geq 0,\ 2/3 - \lambda \geq 0,\ \frac{1}{3} + \lambda + \frac{2}{3} - \lambda \leq 1$

$\Rightarrow v \pm 1/3\, w \in P$ $\qquad \lambda \geq -1/3,\quad 2/3 \geq \lambda,\quad 1 \geq 1$

(3) $\Rightarrow$ (2) (same as "not (2)" $\Rightarrow$ "not (3)")

If $v, v+w, v-w$ all belong to P, then for any vector $c \in \mathbb{R}^n$, we have one of the following:

- $c^T w > 0 \Rightarrow c^T(v+w) = c^T v + c^T w > c^T v$ ($v+w$ higher value)
- $c^T w = 0 \Rightarrow c^T(v+w) = c^T v + c^T w = c^T v$ ($v+w$ same value)
- $c^T w < 0 \Rightarrow c^T(v-w) = c^T v - c^T w > c^T v$ ($v-w$ higher value)

(1) $\Rightarrow$ (3) Take $c = \sum_{i \in I} a_i$ where $I = \{i : a_i^T v = b_i\}$

(You check!)

Condition (1) gives a method for finding all vertices of $P = \{x \in \mathbb{R}^n : a_1^T x \leq b_1, \dots, a_m^T x \leq b_m\}$:

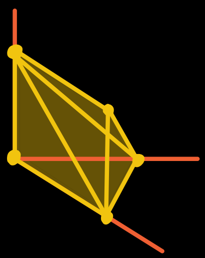
For each subset $\{i_1, \dots, i_n\}$ of $\{1, \dots, m\}$ $\leftarrow$ total # $= \binom{m}{n}$

- check if $a_{i_1}, \dots, a_{i_n}$ linearly independent (no? STOP)
- Compute unique solution v to $a_{i_1}^T x = b_{i_1}, \dots, a_{i_n}^T x = b_{i_n}$
- test whether $v \in P$ (i.e. $a_1^T v \leq b_1, \dots, a_m^T v \leq b_m$)

$v \in P \rightsquigarrow$ vertex $\qquad v \notin P \rightsquigarrow$ not vertex

Ex: $P = \{(x,y,z) \in \mathbb{R}^3 : \underset{(1)}{x \geq 0}, \underset{(2)}{y \geq 0}, \underset{(3)}{z \geq 0}, \underset{(4)}{x+y \leq 1}, \underset{(5)}{x+z \leq 1}, \underset{(6)}{y+z \leq 1}\}$

Equality in



$\{1,2,3\} \to x=0, y=0, z=0 \to (x,y,z) = (0,0,0)$ vertex

$\{1,2,4\} \to x=0, y=0, x+y=1$ infeasible!

$\{1,2,5\} \to x=0, y=0, x+z=1 \to (x,y,z) = (0,0,1)$ vertex

$\{1,2,6\} \to x=0, y=0, y+z=1 \to (x,y,z) = (0,0,1)$ repeat

$\vdots$

all vertices must appear on this list of size $\binom{6}{3} = 20$

$\{1,4,5\} \to x=0, x+y=1, x+z=1 \to (x,y,z) = (0,1,1)$ fails $y+z \leq 1$ not a vertex

$\vdots$

$\{4,5,6\} \to x+y=1, x+z=1, y+z=1 \to (x,y,z) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ vertex

List of vertices: $(0,0,0), (1,0,0), (0,1,0), (0,0,1), (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

The point $v = (0, \frac{1}{2}, \frac{1}{2})$ is not a vertex of P

Inequalities at equality? (1) and (6)

Find $w \in \mathbb{R}^3$ satisfying $x=0, y+z=0$, e.g. $w = (0,1,-1)$

Then for some $\varepsilon > 0$, $v \pm \varepsilon w \in P$

$v + \lambda w = (0, \frac{1}{2}+\lambda, \frac{1}{2}-\lambda)$ (automatically satisfies (1) and (6))

(2) $\frac{1}{2} + \lambda \geq 0 \Rightarrow \lambda \geq -\frac{1}{2}$ (4) $0 + \frac{1}{2} + \lambda \leq 1 \Rightarrow \lambda \leq \frac{1}{2}$

(3) $\frac{1}{2} - \lambda \geq 0 \Rightarrow \lambda \leq \frac{1}{2}$ (5) $0 + \frac{1}{2} - \lambda \leq 1 \Rightarrow -\frac{1}{2} \leq \lambda$

$\Rightarrow v + \lambda w$ feasible for $-\frac{1}{2} \leq \lambda \leq \frac{1}{2}$