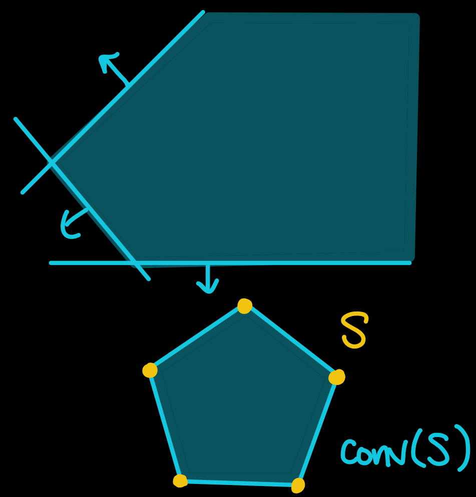
Math 407 – Linear Optimization

Today: Vertices of Polytopes & Polyhedra

Hwk 1 due Oct 5

From last time:



Polyhedron $= \{x \in \mathbb{R}^n : Ax \leq b\}$

$=$ feas. set of an LP

Polytope $=$ convex hull of a finite set of pts in $\mathbb{R}^n$

<u>Prop</u>: For $c \in \mathbb{R}^n$ and $S \subseteq \mathbb{R}^n$ finite,

$$\max\{c^T x : x \in S\} = \max\{c^T x : x \in \mathrm{conv}(S)\}$$

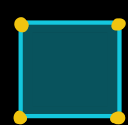
to maximize $c^T x$ over $\mathrm{conv}(S)$, only need to compare <u>$c^T x$ for $x \in S$</u>!

finitely many values

Thm: Every polytope is a polyhedron.

Ex: $S = \{(0,0), (0,1), (1,0), (1,1)\}$

$\mathrm{Conv}(S) = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq 1\}$

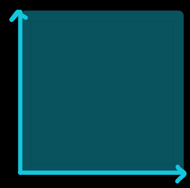


For more: see "V- vs. H-reprensentations" of a polytope

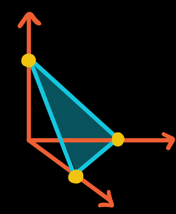
(V=vertex) (H = halfspace)

Thm: Every bounded polyhedron is a polytope, namely, it is the convex hull of its <u>vertices</u>.

Non-ex: $\{(x_1,x_2)\in\mathbb{R}^2 : x_1\geq 0,\ x_2\geq 0\}$ is a polyhedron but not a polytope (not bounded)



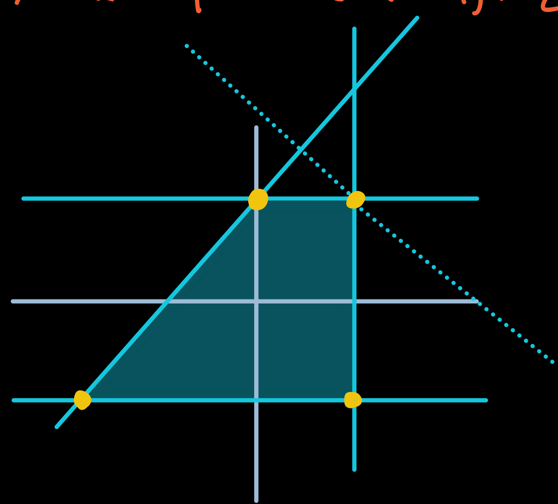
Ex: $\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1\geq 0,\ x_2\geq 0,\ x_3\geq 0,\ x_1+x_2+x_3=1\}$
$= \text{conv}\{(1,0,0),(0,1,0),(0,0,1)\}$



Questions for discussion:

- How should we define the vertex of a polyhedron P?
- Given an inequality description $Ax\leq b$ of P how do we know if a point is a vertex?

Ex 1: $P=\{(x_1,x_2)\in\mathbb{R}^2 : -1\leq x_2\leq 1,\ x_2-1\leq x_1\leq 1\}$



Ex 1': $P\cap\{x : 2x_1\leq 2\}$

Ex 2: $P\cap\{x : x_1+x_2\leq 2\}$

All are equal to conv(S)

$S=\{(0,1),(1,1),(1,-1),(-2,-1)\}$

Some observations:

A vertex satisfies some defining inequalities with equality ($a_i^T v = b_i$ for some i)

Needs to satisfy "enough" inequalities at equality

What is enough? At least n?

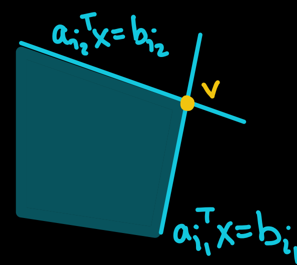
In Ex 1', $v=(1,0)$ satisfies 2 ineq. at equality (namely $x_1 \le 1$ and $2x_1 \le 2$) but isn't a vertex

Problem? $x_1 = 1$, $2x_1 = 2$ not <u>linearly independent</u>

<u>Ideas</u> (union from both sections!)

Given $P = \{x \in \mathbb{R}^n : a_1^T x \le b_1, \ldots, a_m^T x \le b_m\}$ ...

Idea 1) $v \in \mathbb{R}^n$ is a vertex of P if it satisfies all the inequalities ($a_i^T v \le b_i$ for all i) and there are n linearly independent inequalities that hold with equality at v ($a_{i_1}^T v = b_{i_1}, \ldots, a_{i_n}^T v = b_{i_n}$ where $a_{i_1}, \ldots, a_{i_n}$ are linearly independent)



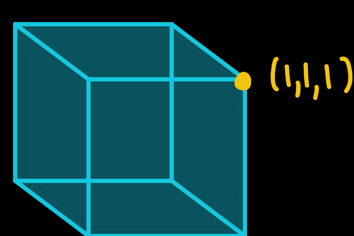
Note: If $a_{i_1}, \ldots, a_{i_n}$ are linearly independent then $x = v$ is the unique solution to $a_{i_1}^T x = b_{i_1}, \ldots, a_{i_n}^T x = b_{i_n}$

Ex: $P = \{x \in \mathbb{R}^3 : 0 \le x_1 \le 1, 0 \le x_2 \le 1, 0 \le x_3 \le 1\}$

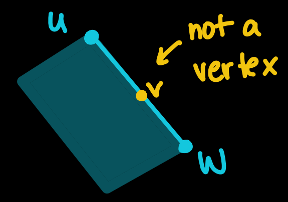
$v = (1,1,1)$ satisfies all ineq ↑

and $\underbrace{x_1 = 1, x_2 = 1, x_3 = 1}_{}$

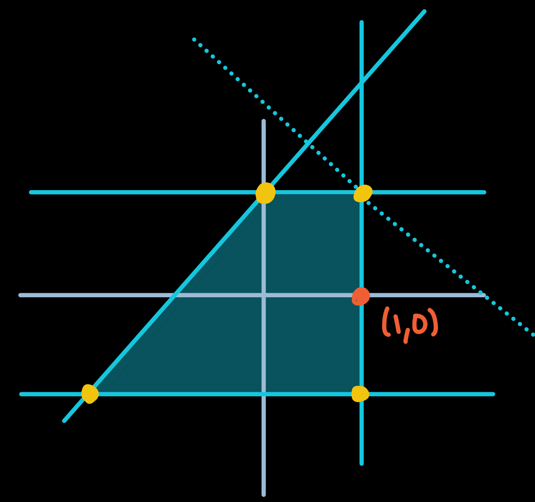
$n=3$ linearly indep. equations



Idea 2) $v \in P$ is <u>not</u> a vertex if it lies on the line segment between two other points in P $\left(v = \lambda u + (1-\lambda) w \text{ with } u, w \in P \setminus \{v\},\ \lambda \in [0,1]\right)$

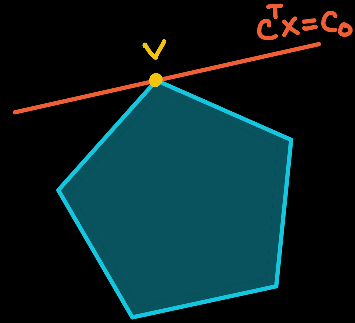


In Ex 1) $v = (1,0)$ is not a vertex
$v = \frac{1}{2}u + \frac{1}{2}w$ where $u = (1,1)$, $w = (1,-1)$
are both pts in P with $u \neq v$, $w \neq v$.



Idea 3) $v \in P$ is a vertex if there is some hyperplane that isolates it from P
That is, there is some $c \in \mathbb{R}^n$, $c_0 \in \mathbb{R}$ s.t.

$$P \cap \{x : c^T x = c_0\} = v$$



In Ex 1) $v = (1,1)$ is a vertex and the hyperplane $c^T x = c_0$ isolates it from the rest of P
$\{(x_1, x_2) : x_1 + x_2 = 2\} \cap P = \{(1,1)\}$