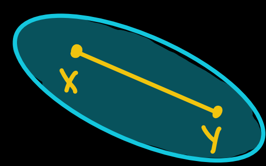
Math 407 – Linear Optimization

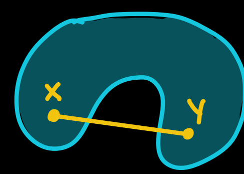
Today: Convexity, Polyhedra, & Polytopes

Convexity Basics

A set $P \subseteq \mathbb{R}^n$ is convex if for every $x, y \in P$ and $\lambda \in [0,1]$, $\lambda x + (1-\lambda) y \in P$
(i.e. P contains the line segment between any two of its points)

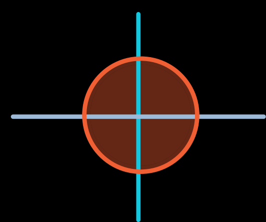


Convex



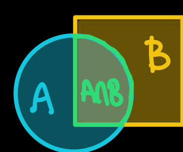
not convex

Ex: $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$

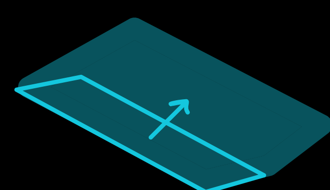


Prop: The intersection of convex sets is convex.

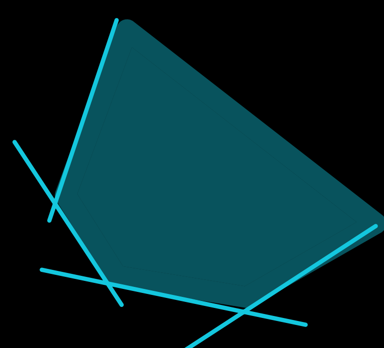
Why? A, B convex, $x, y \in A \cap B$, $\lambda \in [0,1]$
$\Rightarrow \lambda x + (1-\lambda) y$ belongs to A and to B
(by convexity of A, B) $\Rightarrow$ belongs to $A \cap B$.



A halfspace in $\mathbb{R}^n$ is a set of the form $\{x \in \mathbb{R}^n : a^T x \le b\}$ where $a \in \mathbb{R}^n \setminus \{0\}$, $b \in \mathbb{R}$.



A polyhedron is the intersection of finitely many half-spaces.
(= feasible regions of LPs)



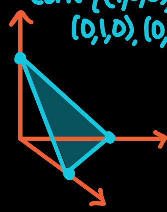
Cor: Any polyhedron is convex.

A <u>convex combination</u> of points $p_1, \dots, p_s \in \mathbb{R}^n$ has the form $\lambda_1 p_1 + \lambda_2 p_2 + \dots + \lambda_s p_s$ where $\lambda_1, \lambda_2, \dots, \lambda_s \geq 0$ and $\sum_{i=1}^{s} \lambda_i = 1$.

The <u>convex hull</u> of a set S is

$$\mathrm{conv}(S) = \left\{ \lambda_1 p_1 + \dots + \lambda_k p_k : \begin{array}{l} k \in \mathbb{N},\ p_1, \dots, p_k \in S \\ \lambda_1, \dots, \lambda_k \geq 0,\ \sum_{i=1}^{s} \lambda_i = 1 \end{array} \right\}$$

conv{(1,0,0), (0,1,0), (0,0,1)}



taking $k=1$ shows $S \subseteq \mathrm{conv}(S)$

Prop: For any $c \in \mathbb{R}^n$ any finite set $S \subseteq \mathbb{R}^n$

$$\max\{c^T x \text{ s.t. } x \in S\} = \max\{c^T x \text{ s.t. } x \in \mathrm{conv}(S)\}$$

(Proof) ($\leq$) follows from $S \subseteq \mathrm{conv}(S)$

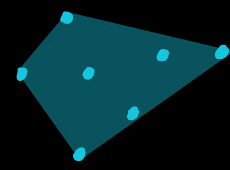
($\geq$) Suppose $m = \max\{c^T x : x \in S\}$ and let $\tilde{x} \in \mathrm{conv}(S)$.

By def, $\tilde{x} = \lambda_1 p_1 + \dots + \lambda_k p_k$ for some $p_1, \dots, p_k \in S$, $\lambda_1, \dots, \lambda_k \geq 0$ with $\sum \lambda_i = 1$. Then

$$\begin{aligned} c^T \tilde{x} = c^T(\lambda_1 p_1 + \dots + \lambda_k p_k) &= \lambda_1 c^T p_1 + \dots + \lambda_k c^T p_k \\ &\leq \lambda_1 m + \dots + \lambda_k m \\ &= (\lambda_1 + \dots + \lambda_k) m = m \end{aligned}$$

using $\lambda_i \geq 0$ and $c^T x \leq m$ for $x \in S$

A <u>polytope</u> is the convex hull of a finite set of points.



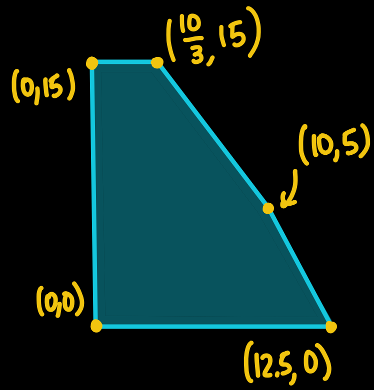
Ex (scarves & hats from last time)

max $10s+6h$ s.t.

$0 \le s \le 15$, $0 \le h \le 15$

$6s+4h \le 80$,

$200s+100h \le 2500$



Feasible region equals the convex hull of 5 pts.

Given constraints, optimal production will be one of only five possibilities, regardless of what profit per item is!