Math 407 – Linear Optimization
Today: Terminology and solving graphically in 2D
HW1 due Thurs. Oct. 5

Recall: A linear optimization problem or linear program (LP) can be written

$$\max\ c^T x \quad \text{s.t.} \quad Ax \le b$$

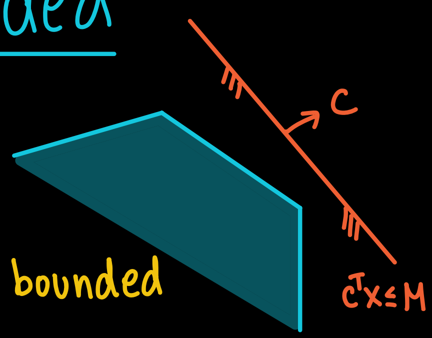
where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$.

means each coord of Ax $\le$ each coord of b

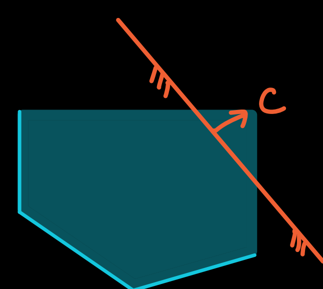
Feasible region: $\{x \in \mathbb{R}^n : Ax \le b\}$

We say this LP is <u>feasible</u> if the feasible region is nonempty and infeasible otherwise.

The LP is <u>bounded</u> if the objective function is bounded from above on the feasible region, i.e. $\exists M \in \mathbb{R}$ s.t. $c^T x \le M$ for all x with $Ax \le b$. Otherwise it is <u>unbounded</u>

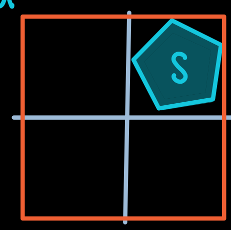


LP bounded

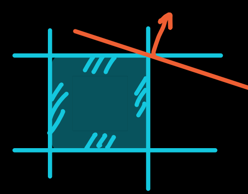


LP unbounded

A set $S \subseteq \mathbb{R}^n$ is <u>bounded</u> if it is contained in some box $\{x \in \mathbb{R}^n : -M \leq x_i \leq M \text{ for all } i=1,\dots,n\}$. Otherwise it is <u>unbounded</u>.



Ex: max x_1+5x_2 s.t. $0 \leq x_1 \leq 1$,
$0 \leq x_2 \leq 1$

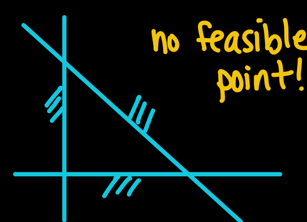


$(x_1,x_2)=(0,0)$ satisfies $\Rightarrow$ feasible

$0 \leq x_1 \leq 1$ and $0 \leq x_2 \leq 1$ $\Rightarrow$ $x_1+5x_2 \leq 6 \Rightarrow$ LP bounded

$0 \leq x_1 \leq 1$ and $0 \leq x_2 \leq 1$ $\Rightarrow$ $|x_1| \leq 1$ and $x_2 \leq 1 \Rightarrow$ feas. region is bounded

Ex: max x_1+5x_2 s.t. $x_1 \leq 0$, $x_2 \leq 0$,
$x_1+x_2 \geq 1$



$x_1 \leq 0$ and $x_2 \leq 0$ implies $x_1+x_2 \leq 0$ $\Rightarrow$ infeasible

$x_1+5x_2 \leq 0$ for all x in feas. region $\Rightarrow$ LP bounded

$|x_1| \leq 1$, $|x_2| \leq 1$ " " $\Rightarrow$ bounded feas. region

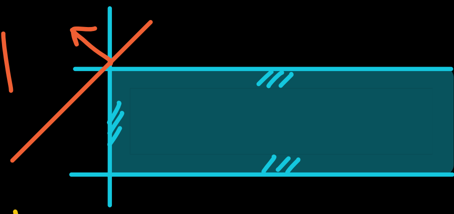
Ex: max x_1+5x_2 s.t. $0 \leq x_1$, $0 \leq x_2 \leq 1$

$(x_1,x_2)=(0,0)$ satisfies $\Rightarrow$ feasible

$(x_1,x_2)=(t,0)$ feasible for all $t \geq 0$ $\Rightarrow$ LP unbounded

and $x_1+5x_2 = t \to \infty$ as $t \to \infty$ and feas. region unbounded

Ex: max $-x_1+x_2$ s.t. $0\le x_1$, $0\le x_2\le 1$



Same feas region $\Rightarrow$ LP feasible,
feas. region unbounded

$-x_1\le 0,\ x_2\le 1 \Rightarrow -x_1+x_2\le 1 \Rightarrow$ LP is bounded

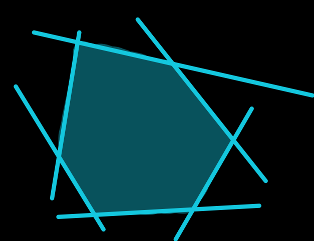
Subtlety: LP can be bounded even when its feasible region is unbounded.

Note: if the feasible region is bounded, then so is the LP:

$$-M\le x_i\le M \text{ for all } i \Rightarrow c_i x_i \le |c_i|\cdot M \quad \text{for all } i=1,\dots,n$$
$$\Rightarrow c_1x_1+\dots+c_nx_n \le (|c_1|+\dots+|c_n|)M$$

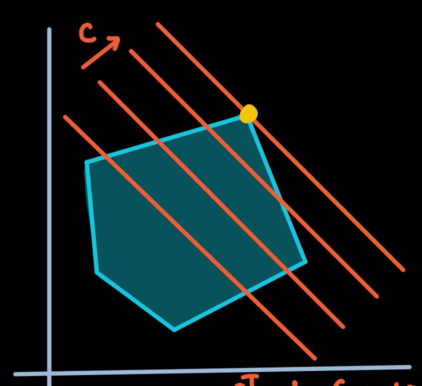
2-dimensional LP's

max $c_1x_1+c_2x_2$ s.t. $a_{11}x_1+a_{12}x_2\le b_1$
$$\vdots$$
$a_{m1}x_1+a_{m2}x_2\le b_m$



Solve graphically:

- Plot feasible region
- Move line with normal vector c as far as possible while it intersects the feasible region



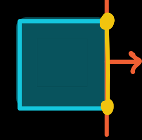
$c^Tx=t$ for varying t

Geometric observation: If the feasible region is bounded optimal value is acheived by a corner point of feasible region ($a_{i1}x_1 + a_{i2}x_2 = b_i$ for ≥ 2 values of i)

we will prove a more general version of this later!

Remark: there can be infinitey many optimal solutions, but at least one will be a corner point

Ex: max x_1 s.t. $0 \leq x_1 \leq 1$, $0 \leq x_2 \leq 1$



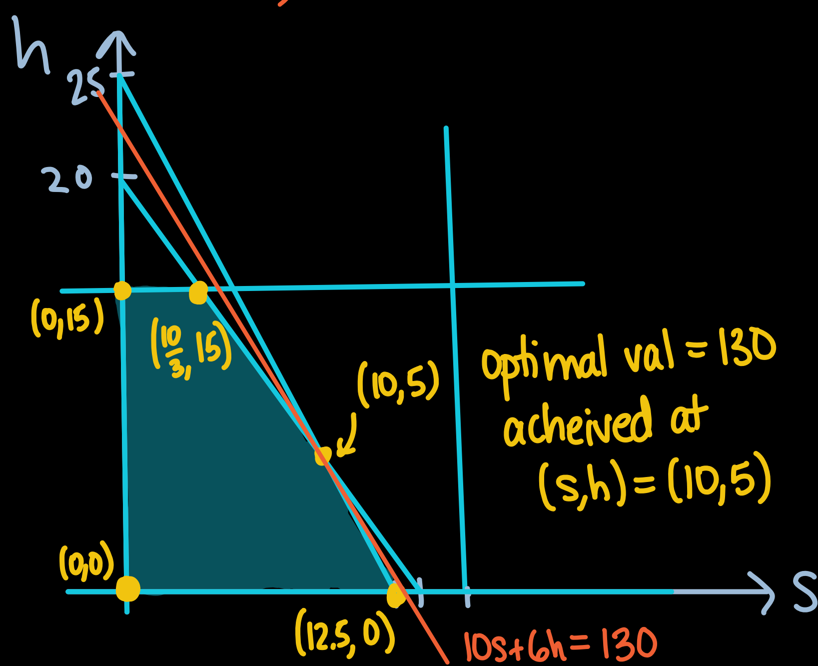
Ex (scarves & hats from last time)

max $10s + 6h$ s.t.

$0 \leq s \leq 15$, $0 \leq h \leq 15$

$6s + 4h \leq 80$,

$200s + 100h \leq 2500$



Finding (some) corner pts:

$h = 15$, $6s + 4h = 80 \Rightarrow (s,h) = (10/3, 15)$ (feasible) $10s + 6h \approx 123.3$

$6s + 4h = 80$, $200s + 100h = 2500 \Rightarrow (s,h) = (10,5)$ (feasible) $10s + 6h = 130$

$h = 0$, $200s + 100h = 2500 \Rightarrow (s,h) = (12.5, 0)$ (feasible) $10s + 6h = 125$