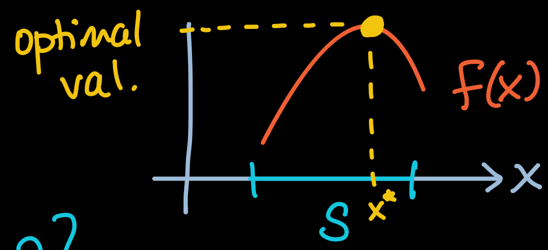
# Math 407 - Linear Optimization

## Today: Course logistics and introduction

Optimization problem: maximize $f(x)$ s.t. $x \in S$

$f(x)$ = "objective function"

$S$ = feasible set/region




optimal value = $\max\{f(x) \text{ s.t. } x \in S\}$

optimal solution = element $x^* \in S$ s.t. $f(x^*) =$ opt. value

In a linear optimization problem,

$f(x) = c_1x_1 + \ldots + c_nx_n$ is linear $\quad c = (c_1, \ldots, c_n)$ = "cost vector"

$S$ = set of solutions to finitely-many affine-linear equations and inequalities

Note: we can always rewrite constraints as a system of inequalities of the form $a^Tx \le b$.

$a^Tx \ge b \iff -a^Tx \le -b$

$a^Tx = b \iff a^Tx \le b$ and $a^Tx \ge b$

$\Rightarrow$ we can write $S$ as

$$S = \left\{ x \in \mathbb{R}^n : \underbrace{\begin{matrix} a_{11}x_1 + \ldots + a_{1n}x_n \le b_1 \\ \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n \le b_m \end{matrix}}_{} \right\}$$

written as $Ax \le b$ for $A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix}$ $\quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

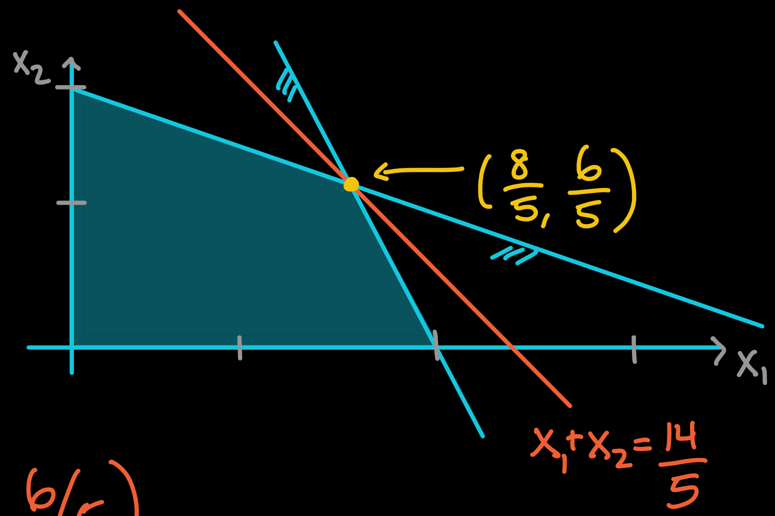
Ex: max $x_1 + x_2$ s.t.

$x_1 \geq 0$, $x_2 \geq 0$, $x_1 + 2x_2 \leq 4$, $3x_1 + x_2 \leq 6$

$-x_1 \leq 0$ $\quad$ $-x_2 \leq 0$

Compact form:

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 4 \\ 6 \end{pmatrix}$$




max value $= 14/5$
acheived at $x = (8/5, 6/5)$

## Why linear optimization?

- efficient algorithms (in theory and in practice)
  main alg for this course: Simplex Method
  (developed by Dantzig 1947 at US Air Force)
  Other algorithms: "ellipsoid", "interior point"
- Powerful modeling tool
  Many real-world opt. problems can be formulated (and solved) as LPs!

E.g. resource allocation, scheduling, optimal transport, and many more!

Ex: A small business makes hats and scarves from imported wool

| | time (hrs/item) | wool (g/item) | demand (items/week) |
|---|---|---|---|
| scarf | 6 | 200 | 15 |
| hat | 4 | 100 | 15 |

Total employee work time: $\leq 80$ hrs/week
Total wool available: 2500 g/week
If they make \$10 profit per scarf and \$6 per hat how many scarves/hats should they make per week to maximize profit?

Modelled as a linear opt. problem:

Variables: $s = \#$ scarves  $h = \#$ hats
Objective: maximize $10s + 6h$ (profit)

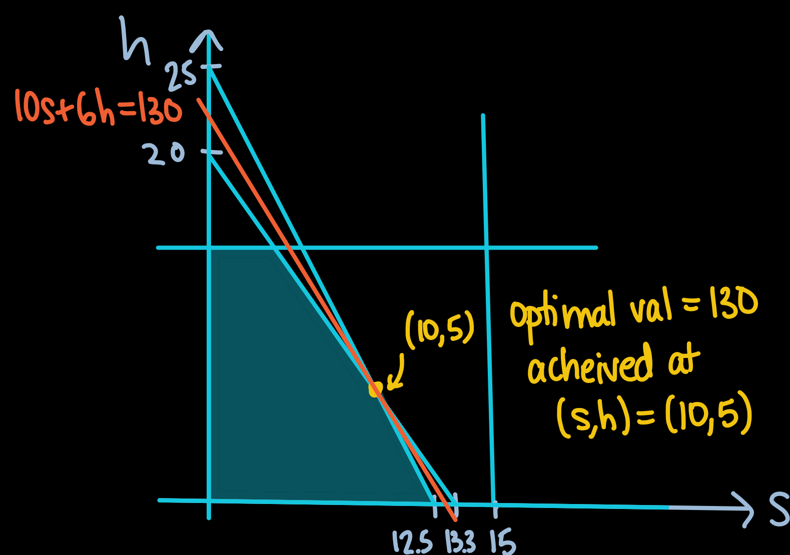
Constraints:

$0 \leq s \leq 15$

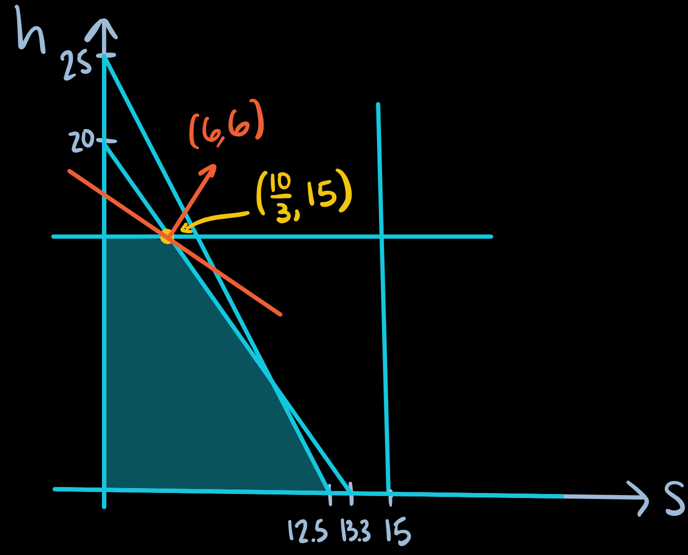
$0 \leq h \leq 15$

$6s + 4h \leq 80$

$200s + 100h \leq 2500$



If instead they make \$6 profit on either,
↳ objective function becomes $6s+6h$




Maximum acheived by $(s,h) = (10/3, 15)$

## Goals for the course

To understand...

- the geometry of feasible sets and optimal solutions
- algebraic certificates of optimality
- the mechanics of the simplex method
- modelling (formulating real world problems as linear opt. problems)