Large Example of the Simplex Method

UW Math 407, Fall 2022

Original LP: $\max x_1 + x_2$ such that $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$ where

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 2 \\ -3 \\ -1 \\ -1 \end{pmatrix}$$

In order to set up the auxiliary LP, we multiply the 2nd, 3rd, and 4th equations by -1, to make $\mathbf{b} \geq 0$, re-writing the original LP as

$\max x_1 + x_2$ such that $\tilde{A}\mathbf{x} = \tilde{\mathbf{b}}$ and $\mathbf{x} \geq 0$ where

$$\tilde{A} = \begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{\mathbf{b}} = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 1 \end{pmatrix}$$

PHASE I

The auxiliary linear program is
$\max -x_9 - x_{10} - x_{11} - x_{12}$ such that

$$\tilde{A}\mathbf{x} + \begin{pmatrix} x_9 \\ x_{10} \\ x_{11} \\ x_{12} \end{pmatrix} = \tilde{\mathbf{b}}, \quad \mathbf{x} \geq 0, \quad \text{and} \quad \begin{pmatrix} x_9 \\ x_{10} \\ x_{11} \\ x_{12} \end{pmatrix} \geq 0$$

The "obvious" feasible basis is $\{9, 10, 11, 12\}$, from which we can run the simplex method to solve the auxiliary linear program:

$B = \{9, 10, 11, 12\}$, $\mathcal{T}(B)$:

$$\begin{aligned} x_9 &= 2 + x_1 - x_5 - x_6 - x_7 \\ x_{10} &= 3 + x_1 - x_2 + x_8 \\ x_{11} &= 1 - x_2 - x_3 + x_6 \\ x_{12} &= 1 + x_3 - x_4 + x_7 \\ z &= -7 - 2x_1 + 2x_2 + x_4 + x_5 - x_8 \end{aligned}$$

We can choose x_2, x_4, or x_5 to enter. Let's choose x_2. With $x_1 = x_3 = x_4 = x_5 = x_6 = x_7 = x_8 = 0$, we find that

$$\begin{aligned} x_9 &= 2 \\ x_{10} &= 3 - x_2 \\ x_{11} &= 1 - x_2 \\ x_{12} &= 1 \end{aligned}$$

The tightest constraint is $x_2 \leq 1$, coming from $x_{11} \geq 0$, so x_{11} leaves.

By hand, we can solve the "$x_{11} =$" equation for x_2 and use that to compute the simplex tableau for the new basis:

$B = \{2, 9, 10, 12\}$, $\mathcal{T}(B)$:

$$\begin{aligned}
x_2 &= 1 - x_3 + x_6 - x_{11} \\
x_9 &= 2 + x_1 - x_5 - x_6 - x_7 \\
x_{10} &= 2 + x_1 + x_3 - x_6 + x_8 + x_{11} \\
x_{12} &= 1 + x_3 - x_4 + x_7 \\
z &= -5 - 2x_1 - 2x_3 + x_4 + x_5 + 2x_6 - x_8 - 2x_{11}
\end{aligned}$$

We can choose x_4, x_5, or x_6 to enter. Let's choose x_6. With $x_1 = x_3 = x_4 = x_5 = x_7 = x_8 = x_{11} = 0$, we find that

$$\begin{aligned}
x_2 &= 1 + x_6 \\
x_9 &= 2 - x_6 \\
x_{10} &= 2 - x_6 \\
x_{12} &= 1
\end{aligned}$$

The tightest constraint is $x_6 \leq 2$, coming from both $x_9 \geq 0$ and $x_{10} \geq 0$, so we can choose either x_9 or x_{10} to leave. Let's choose x_9. Solving for the equation"$x_9 =$" in the tableau above for x_2 let's us compute the next simplex tableau:

$B = \{2, 6, 10, 12\}$, $\mathcal{T}(B)$:

$$\begin{aligned}
x_2 &= 3 + x_1 - x_3 - x_5 - x_7 - x_9 - x_{11} \\
x_6 &= 2 + x_1 - x_5 - x_7 - x_9 \\
x_{10} &= x_3 + x_5 + x_7 + x_8 + x_9 + x_{11} \\
x_{12} &= 1 + x_3 - x_4 + x_7 \\
z &= -1 - 2x_3 + x_4 - x_5 - 2x_7 - x_8 - 2x_9 - 2x_{11}
\end{aligned}$$

We can only choose x_4 to enter. With $x_1 = x_3 = x_5 = x_7 = x_8 = x_9 = x_{11} = 0$, we find that

$$\begin{aligned}
x_2 &= 3 \\
x_6 &= 2 \\
x_{10} &= 0 \\
x_{12} &= 1 - x_4
\end{aligned}$$

The tightest constraint is $x_4 \leq 1$, coming from $x_{12} \geq 0$, so x_{12} leaves. (Subtlety: x_{10} is also zero at this point, but x_{10} cannot leave. Indeed, $\{2, 4, 6, 12\}$ is not a basis!)

$B = \{2, 4, 6, 10\}$, $\mathcal{T}(B)$:

$$\begin{aligned} x_2 &= 3 + x_1 - x_3 - x_5 - x_7 - x_9 - x_{11} \\ x_4 &= 1 + x_3 + x_7 - x_{12} \\ x_6 &= 2 + x_1 - x_5 - x_7 - x_9 \\ x_{10} &= x_3 + x_5 + x_7 + x_8 + x_9 + x_{11} \\ z &= -x_3 - x_5 - x_7 - x_8 - 2x_9 - 2x_{11} - x_{12} \end{aligned}$$

There are no positive coefficients and so we are done! This certifies that the optimal value of the auxiliary linear program is zero and so the original linear program is feasible.
The corresponding basic feasible solution of the auxiliary LP is given by
$x_1 = x_3 = x_5 = x_7 = x_8 = x_9 = x_{11} = x_{12} = 0$ and $x_2 = 3$, $x_4 = 1$, $x_6 = 2$, and $x_{10} = 0$.
That is, $(x_1, \ldots, x_{12}) = (0, 3, 0, 1, 0, 2, 0, 0, 0, 0, 0, 0)$. Dropping the x_9, x_{10}, x_{11}, x_{12} coordinates gives a feasible solution of the original linear program, namely
$(x_1, \ldots, x_8) = (0, 3, 0, 1, 0, 2, 0, 0)$.

Subtlety: the solution to the auxiliary LP is represented by more than one basis. In addition to $\{2, 4, 6, 10\}$, it is also represented by $\{2, 3, 4, 6\}$ (among others). If we use the equation $x_{10} = x_3 + x_5 + x_7 + x_8 + x_9 + x_{11}$ in the tableau above to solve for x_3, we obtain the simplex tableau:
$B = \{2, 3, 4, 6\}$, $\mathcal{T}(B)$:

$$\begin{aligned} x_2 &= 3 + x_1 + x_8 - x_{10} \\ x_3 &= -x_5 - x_7 - x_8 - x_9 + x_{10} - x_{11} \\ x_4 &= 1 - x_5 - x_8 - x_9 + x_{10} - x_{11} - x_{12} \\ x_6 &= 2 + x_1 - x_5 - x_7 - x_9 \\ z &= -x_9 - x_{10} - x_{11} - x_{12} \end{aligned}$$

This corresponds to the same point. (The benefit is that $\{2, 3, 4, 6\}$ will be a feasible basis for the original LP.)

PHASE II

Now we can remove the variables $x_9, x_{10}, x_{11}, x_{12}$ to find a feasible basis and tableau for the original linear program:

$B = \{2, 3, 4, 6\}$, $\mathcal{T}(B)$:

$$\begin{aligned} x_2 &= 3 + x_1 + x_8 \\ x_3 &= -x_5 - x_7 - x_8 \\ x_4 &= 1 - x_5 - x_8 \\ x_6 &= 2 + x_1 - x_5 - x_7 \end{aligned}$$

Now we can remember the original objective function! And starting from this feasible basis, use the simplex method to maximize it.

$B = \{2, 3, 4, 6\}$, $\mathcal{T}(B)$:

$$\begin{aligned} x_2 &= 3 + x_1 + x_8 \\ x_3 &= -x_5 - x_7 - x_8 \\ x_4 &= 1 - x_5 - x_8 \\ x_6 &= 2 + x_1 - x_5 - x_7 \\ z &= 3 + 2x_1 + x_8 \end{aligned}$$

We can choose x_1 or x_8 to enter. Let's choose x_1. Taking $x_5 = x_7 = x_8 = 0$, we find that

$$\begin{aligned} x_2 &= 3 + x_1 \\ x_3 &= 0 \\ x_4 &= 1 \\ x_6 &= 2 + x_1 \\ z &= 3 + 2x_1. \end{aligned}$$

From this we see that for every nonnegative value of x_1, the coordinates x_2, x_3, x_4, x_6 are nonnegative. That is, for every $x_1 \geq 0$, the point $\mathbf{x} = (x_1, 3 + x_1, 0, 1, 0, 2 + x_1, 0, 0)$ is feasible and has objective function value $3 + 2x_1$. By making x_1 arbitrarily large, we can make the objective function arbitrarily large, so the original linear program is unbounded.