Math 407 – Homework 6

Due on Thursday, Nov. 16

You are welcome to talk with other students in the class about problems but should write up solutions on your own. Solutions can be handwritten or typed but need to be legible and submitted via Gradescope by the end of the day on Thursday. You should justify all your answers in order to receive full credit.

Problem 1. For each of the pairs $\mathbf{c}$, $\mathbf{v}$ below, decide: is $\mathbf{v}$ is an optimal solution of the linear program $\max \mathbf{c}^T\mathbf{x}$ s.t. $A\mathbf{x} \leq \mathbf{b}$ where

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}?$$

If yes, use complimentary slackness to find a dual solution $\mathbf{y} \in \mathbb{R}^5$ that certifies optimality. If not, justify why no such $\mathbf{y}$ exists.

(a) $\mathbf{c} = (4, 1, 2)$ $\quad\mathbf{v} = (1, 0, 1)$
(b) $\mathbf{c} = (1, 0, 1)$ $\quad\mathbf{v} = (1, 1, 0)$
(c) $\mathbf{c} = (-2, 1, 1)$, $\quad\mathbf{v} = (0, 1, 1)$
(d) $\mathbf{c} = (1, 0, 1)$ $\quad\mathbf{v} = (0, 0, 1)$
(e) $\mathbf{c} = (-2, -1, -1)$ $\quad\mathbf{v} = (0, 0, 0)$

In Problem 3(a) of Homework 5, you showed that the following are primal-dual linear programs

$$\text{(P)} \qquad \max\ \mathbf{c}^T\mathbf{x} \ \text{ s.t. } \ A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq 0$$

$$\text{(D)} \qquad \min\ \mathbf{b}^T\mathbf{y} \ \text{ s.t. } \ A^T\mathbf{y} - \mathbf{s} = \mathbf{c},\ \mathbf{s} \geq 0$$

where $A \in \mathbb{R}^{m\times n}$ has rank m, $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c} \in \mathbb{R}^n$. We will use this primal-dual pair for Problems 2 and 3.

Problem 2. (Complimentary Slackness.)

(a) Let $\mathbf{x}$ be feasible for (P) and $(\mathbf{y}, \mathbf{s})$ be feasible for (D). Show that $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$ if and only if $x_i s_i = 0$ for all $i = 1, \ldots, n$.

(b) The linear program

$$\text{(P)} \quad \max\ 10x_1 + 14x_2 + 3x_3 \quad \text{s.t.} \quad \begin{aligned} &x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0 \\ &x_1 + x_2 + x_4 = 1 \\ &3x_1 + 2x_2 + x_3 + x_5 = 4 \end{aligned}$$

has optimal solution $\mathbf{x}^* = (0, 1, 2, 0, 0)$. Formulate the dual linear program as above (using variables $y_1, y_2, s_1, s_2, s_3, s_4, s_5$) and use complimentary slackness to find an optimal solution $(\mathbf{y}^*, \mathbf{s}^*)$.

Problem 3. (Equational form for everyone.) Let M be an $(n-m) \times n$ matrix whose (right) kernel equals the column span of A^T and let $\mathbf{d} \in \mathbb{R}^n$ be a vector for which $A\mathbf{d} = \mathbf{b}$.

(a) Show that a point $\mathbf{s} \in \mathbb{R}^n$ satisfies $M\mathbf{s} = -M\mathbf{c}$ if and only if there exists $\mathbf{y} \in \mathbb{R}^m$ for which $A^T\mathbf{y} - \mathbf{s} = \mathbf{c}$ and that, if it exists, the vector $\mathbf{y}$ is unique.

(b) Show that the dual problem (D) is equivalent to the following:

$$(\mathrm{D}') \qquad \min \ \mathbf{d}^T\mathbf{s} + \mathbf{d}^T\mathbf{c} \ \text{ s.t. } \ M\mathbf{s} = -M\mathbf{c}, \ \mathbf{s} \geq 0$$

(This is an LP in equational form with n variables $s_1, \ldots, s_n$ and $n-m$ equations!)

(c) Show that if $A = \begin{pmatrix} -Q & I_m \end{pmatrix}$ where $Q \in \mathbb{R}^{m \times (n-m)}$ and I_m denotes the $m \times m$ identity matrix, then we can take $M = \begin{pmatrix} I_{n-m} & Q^T \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} \mathbf{0}_{n-m} \\ \mathbf{b} \end{pmatrix}$.

(d) Use (a)-(c) to formulate the dual LP you wrote in Problem 2(b) as a linear program in the variables $s_1, \ldots, s_5$ (without y_1, y_2).