# Math 407 – Homework 5

Due on Thursday, November 9

You are welcome to talk with other students in the class about problems but should write up solutions on your own. Solutions can be handwritten or typed but need to be legible and submitted via Gradescope by the end of the day on Thursday. You should justify all your answers in order to receive full credit.

**Problem 1.** Consider the linear program

$$\begin{aligned} \max \quad 3x_1 + x_2 \quad \text{s.t.} \quad & x_1 + x_2 + x_3 = 5 \\ & x_1 - x_2 + 2x_3 \geq 0 \\ & x_1 \geq 0 \\ & x_1 - x_2 \leq 2 \end{aligned}$$

(a) Consider the linear combination

$$y_1(x_1 + x_2 + x_3 = 5) + y_2(x_1 - x_2 + 2x_3 \geq 0) + y_3(x_1 \geq 0) + y_4(x_1 - x_2 \leq 2).$$

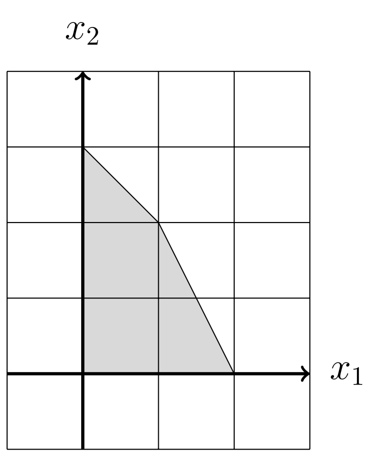
How should $y_1, y_2, y_3, y_4$ be chosen so that this gives an upper bound on $3x_1 + x_2$? What upper bound does it give (as a function of $y_1, y_2, y_3, y_4$)?

(b) Formulate the problem of finding the best such upper bound on $3x_1 + x_2$ as a linear program. (This is the dual linear program.)

(c) I claim that $\mathbf{x}^* = (4, 2, -1)^T$ an optimal solution for the primal linear program and $\mathbf{y}^* = (2, -1, 0, 2)^T$ is an optimal for the dual linear program. What needs to be checked to verify this?

**Problem 2.** Consider the (primal) linear program

$$\max \quad \mathbf{c}^T\mathbf{x} \quad \text{s.t.} \quad x_1 \geq 0,\ x_2 \geq 0,\ 2x_1 + x_2 \leq 4,\ x_1 + x_2 \leq 3$$




(a) For what values of $\mathbf{c}$ is the maximum of this LP attained at $\mathbf{x} = (1, 2)^T$? For what values of $\mathbf{c}$ is the maximum attained uniquely at $\mathbf{x} = (1, 2)^T$?

(b) Pick a cost vector $\mathbf{c}$ for which the maximum is attained uniquely at $\mathbf{x} = (1, 2)^T$, write the dual linear program and draw the feasible region of the dual linear program in the $(y_3, y_4)$-plane (i.e. using $y_3$, $y_4$ as parameters).

(c) Find the optimal solution $\mathbf{y}^* \in \mathbb{R}^4$ of the dual LP. Which coordinates $y_j$ are strictly positive?

**Problem 3** (The dual of a linear program in equational form).
Consider the (primal) linear program in equational form:

$$\text{(P)} \qquad \max \quad \mathbf{c}^T\mathbf{x} \quad \text{s.t.} \quad A\mathbf{x} = \mathbf{b}, \ \mathbf{x} \geq 0$$

where $A \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c} \in \mathbb{R}^n$ and the variable $\mathbf{x}$ ranges over $\mathbb{R}^n$.
The dual linear program is

$$\text{(D)} \qquad \min \quad \mathbf{b}^T\mathbf{y} \quad \text{s.t.} \quad A^T\mathbf{y} - \mathbf{s} = \mathbf{c}, \ \mathbf{s} \geq 0$$

where $\mathbf{y}$ ranges over $\mathbb{R}^m$ and $\mathbf{s}$ ranges over $\mathbb{R}^n$.

(a) Show that for any feasible point $(\mathbf{y}, \mathbf{s})$ of (D), $\mathbf{b}^T\mathbf{y}$ gives an upper bound on the primal linear program (P).
*Hint:* show that $\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$ for any feasible point $\mathbf{x}$ of (P).

(b) Show that the linear program (D) is equivalent to

$$\text{(D}'\text{)} \qquad \min \quad \mathbf{b}^T\mathbf{y} \quad \text{s.t.} \quad A^T\mathbf{y} \geq \mathbf{c}$$