Math 407 – Homework 4

Due on Thursday, October 26

You are welcome to talk with other students in the class about problems but should write up solutions on your own. Solutions can be handwritten or typed but need to be legible and submitted via Gradescope by the end of the day on Thursday. You should justify all your answers in order to receive full credit.

Problem 1. Solve the linear program

$$\max \quad x_2 + 2x_3 \quad \text{s.t.} \quad A\mathbf{x} = \mathbf{b} \quad \text{and} \quad \mathbf{x} \geq 0$$

where

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 3 \end{pmatrix}$$

using the simplex method starting from feasible basis $B = \{4, 5, 6, 7\}$.

At each feasible basis used, give the corresponding simplex tableau and specify the corresponding basic feasible vertex in $\mathbb{R}^7$. At each pivot step, you should explain your choice of entering and leaving variable.

Problem 2. Consider a linear program in $n = 8$ variables and $m = 4$ equations whose simplex tableau $\mathcal{T}(\{5, 6, 7, 8\})$ is

$$\begin{array}{ccccccccccc} x_5 & = & 1 & + & x_1 & + & x_2 & & & + & 2x_4 \\ x_6 & = & 2 & - & x_1 & & & & & + & x_4 \\ x_7 & = & 3 & & & + & 2x_2 & + & x_3 & + & x_4 \\ x_8 & = & 5 & + & 2x_1 & + & x_2 & - & x_3 & & \\ \hline z & = & 8 & + & \lambda x_1 & + & \mu x_2 & - & x_3 & - & 2x_4 \end{array}$$

where $\lambda, \mu \in \mathbb{R}$. Use the simplex method to answer the following:
For what values of $\lambda, \mu \in \mathbb{R}$ is this linear program bounded?

Hint: Think about μ first.

Problem 3 (The set of optimal solutions)**.** Consider a linear program in $n = 6$ variables with $m = 2$ equations for which $\{2,3\}$ is a feasible basis and whose simplex tableau $\mathcal{T}(\{2,3\})$ has last line $z = 8 - x_5 - x_6$.

(a) Show that a feasible point $\mathbf{x} = (x_1, \ldots, x_6) \in \mathbb{R}^6$ is an optimal solution for this linear program if and only if $x_5 = x_6 = 0$.

(b) Describe (using affine-linear equations and inequalities) the set of optimal solutions to the linear program whose simplex tableau $\mathcal{T}(\{2,3\})$ is

$$\begin{array}{rclcccccccccc} x_2 & = & 1 & + & x_1 & - & 7x_4 & + & 3x_5 & - & 2x_6 \\ x_3 & = & 1 & - & x_1 & - & 5x_4 & - & 2x_5 & + & x_6 \\ \hline z & = & 8 & & & & & - & x_5 & - & x_6 \end{array}$$

and draw the set of optimal solutions in a plane parametrized by (x_1, x_4).

(c) Decide whether or not the linear program whose simplex tableau $\mathcal{T}(\{2,3\})$ is

$$\begin{array}{rclcccccccccc} x_2 & = & 1 & + & x_1 & - & 7x_4 & + & 3x_5 & - & 2x_6 \\ x_3 & = & 0 & - & x_1 & - & 5x_4 & - & 2x_5 & + & x_6 \\ \hline z & = & 8 & & & & & - & x_5 & - & x_6 \end{array}$$

has a unique optimal solution.