Math 407 – Homework 3

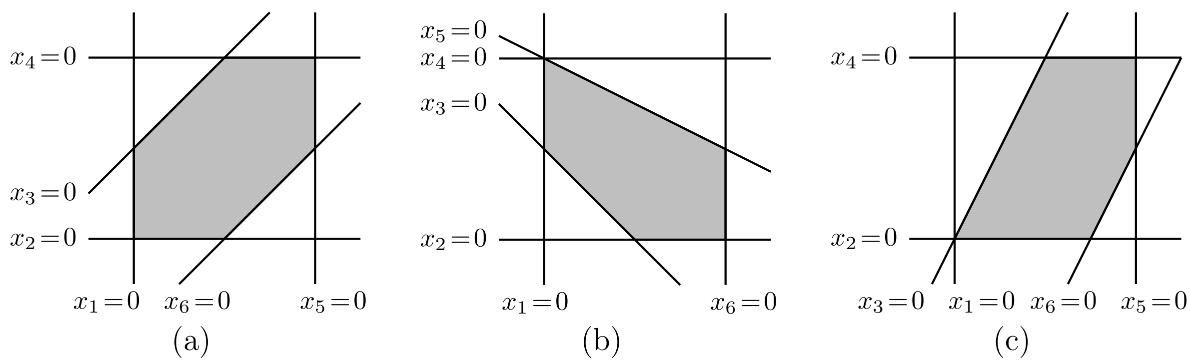
Due on Thursday, October 19

You are welcome to talk with other students in the class about problems but should write up solutions on your own. Solutions can be handwritten or typed but need to be legible and submitted via Gradescope by the end of the day on Thursday. You should justify all your answers in order to receive full credit.

Problem 1. Each of pictures below is the feasible region of a linear program $\max\{\mathbf{c}^T\mathbf{x} : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$ in 6 variables with 4 equations ($n = 6, m = 4$).
For each, answer the following:

- Is the basis $\{3, 4, 5, 6\}$ feasible?
- If so, is there a different basis that corresponds to the same feasible solution?

Remember to explain your answer.



Problem 2. Parts (a) through (d) refer to the linear program

$$\max \quad x_4 + x_5 + x_6 + x_7 \quad \text{s.t.} \quad A\mathbf{x} = \mathbf{b} \quad \text{and} \quad \mathbf{x} \geq 0$$

where

$$A = \begin{pmatrix} 1 & -1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} -2 \\ 2 \\ 4 \\ 3 \end{pmatrix}$$

and the point $\mathbf{v} = (0, 0, 0, 2, 0, 0, 3)^T$.

(a) Is $\mathbf{v}$ a basic feasible solution to this linear program?

(b) Is $B = \{4, 5, 6, 7\}$ a feasible basis?

(c) Reformulate this LP as an LP (not in equational form) involving only x_1, x_2, x_3.
(*Hint:* parametrize $\{\mathbf{x} \in \mathbb{R}^7 : A\mathbf{x} = \mathbf{b}\}$ using the variables x_1, x_2, x_3 as parameters.)

(d) Use your formulation in (c) to show that $\mathbf{v}$ is an optimal solution to this LP.

Problem 3 (Vertex or bust)**.** The proof of Theorem 4.2.3 in MG gives an algorithm that, starting feasible solution $\mathbf{x}_0$ of a linear program $\max\{\mathbf{c}^T\mathbf{x} : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$, produces a basic feasible solution $\tilde{\mathbf{x}}$ with $\mathbf{c}^T\tilde{\mathbf{x}} \geq \mathbf{c}^T\mathbf{x}_0$ or shows that the linear program is unbounded. This is summarized as follows:

$\mathbf{x} := \mathbf{x}_0$.
REPEAT:

(1) If $\mathbf{x}$ is a basic feasible solution, output $\mathbf{x}$ and stop.
(2) Find nonzero vector $\mathbf{w}$ such that $A\mathbf{w} = 0$ and any coordinate that is zero in $\mathbf{x}$ is also zero in $\mathbf{w}$.
(3) If $\mathbf{c}^T\mathbf{w} < 0$, replace $\mathbf{w} := -\mathbf{w}$.
(4) If $\mathbf{c}^T\mathbf{w} > 0$,
  (5) If $\mathbf{x} + \lambda\mathbf{w}$ is feasible for all $\lambda \geq 0$, output "LP is unbounded" and stop.
  (6) Find $\lambda^* = \max\{\lambda \in \mathbb{R} : \mathbf{x} + \lambda\mathbf{w} \text{ is feasible}\}$ and set $\mathbf{x} := \mathbf{x} + \lambda^*\mathbf{w}$.
(7) If $\mathbf{c}^T\mathbf{w} = 0$,
  (8) If $\mathbf{x} + \lambda\mathbf{w}$ is feasible for all $\lambda \geq 0$, replace $\mathbf{w} := -\mathbf{w}$.
  (9) Find $\lambda^* = \max\{\lambda \in \mathbb{R} : \mathbf{x} + \lambda\mathbf{w} \text{ is feasible}\}$ and set $\mathbf{x} := \mathbf{x} + \lambda^*\mathbf{w}$.

Starting from the point $\mathbf{x}_0 = (1, 2, 3, 4, 5, 0, 0)$ and using $\mathbf{w} = (0, 1, 0, 2, -1, 0, 0)$ in the first iteration of (2), follow this algorithm in each of the following LPs. For each, you should either find a basic feasible solution with $\tilde{\mathbf{x}}$ with $\mathbf{c}^T\tilde{\mathbf{x}} \geq \mathbf{c}^T\mathbf{x}_0$ or conclude that the LP is unbounded. Both LPs have the form:

$$\max \quad \mathbf{c}^T\mathbf{x} \quad \text{such that } (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \geq 0, \quad \begin{aligned} x_1 - 2x_2 + x_4 - 2x_6 &= 1 \\ x_2 - 2x_3 + x_5 - 2x_7 &= 1 \\ x_3 - 2x_1 + 2x_6 - x_7 &= 1 \end{aligned}$$

(a) For the first LP, use $\mathbf{c}^T\mathbf{x} = -2x_1 + 3x_2 - 2x_3 - x_4$.
(b) For the second, use $\mathbf{c}^T\mathbf{x} = -2x_1 + 3x_2 + 2x_3 - x_4$.