Math 407 – Homework 2

Due on Thursday, October 12

You are welcome to talk with other students in the class about problems but should write up solutions on your own. Solutions can be handwritten or typed but need to be legible and submitted via Gradescope by the end of the day on Thursday. You should justify all your answers in order to receive full credit.

Problem 1. Find all vertices of the polyhedron of points $(x, y, z) \in \mathbb{R}^3$ satisfying:

$$(1)\ x \geq 0 \qquad (2)\ y \geq 0 \qquad (3)\ z \geq 0 \qquad (4)\ x + 4y \leq 4 \qquad (5)\ x + 2y + 3z \leq 6$$

For each vertex $\mathbf{v}$, write down the subset of $\{1, 2, 3, 4, 5\}$ indexing the set of inequalities for which equality holds at $\mathbf{v}$.

For example, for the polyhedron of points $(x, y) \in \mathbb{R}^2$ satisfying

$$(1)\ x \geq 0 \qquad (2)\ y \geq 0 \qquad (3)\ x + y \leq 1,$$

the analogous answer would be:

- $(0, 0) \leftrightarrow \{1, 2\}$
- $(0, 1) \leftrightarrow \{1, 3\}$
- $(1, 0) \leftrightarrow \{2, 3\}$

You are welcome to use a computer to solve systems of linear equations.

Problem 2. Consider the linear program

$$\begin{aligned}
\max \quad x_1 + x_2 + x_4 + x_7 \quad \text{subject to} \quad & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0, x_7 \geq 0 \\
& x_1 + x_2 \leq 2 \\
& x_1 + x_3 \leq 2 \\
& x_2 + x_4 + x_6 \leq 2 \\
& x_3 + x_4 + x_5 \leq 2 \\
& x_5 + x_7 \leq 2 \\
& x_6 + x_7 \leq 2
\end{aligned}$$

and the point $\mathbf{v} = (1, 1, 1, 1, 0, 0, 2)$.

(a) Find a nonzero vector $\mathbf{w}$ so that $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$ both belong to the feasible region of this LP. (This shows that $\mathbf{v}$ is not a vertex!)

(b) For what values of $\lambda \in \mathbb{R}$ is $\mathbf{v} + \lambda\mathbf{w}$ feasible?

(c) Which of the feasible points $\mathbf{v} + \lambda\mathbf{w}$ (that you found in part (b)) achieves the largest value of the objective function?

Problem 3. Let P be the convex hull of a finite set $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\} \subset \mathbb{R}^n$ and consider the set C_j of all cost vectors that achieving their maximum value over P at $\mathbf{v}_j$. That is,

$$C_j \; = \; \left\{\mathbf{c} \in \mathbb{R}^n \text{ such that } \mathbf{c}^T\mathbf{v}_j = \max\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \in P\}\right\},$$

(a) For the specific choice of points $\mathbf{v}_1 = (0,0)$, $\mathbf{v}_2 = (1,0)$, and $\mathbf{v}_3 = (0,1)$, describe the set C_2 of cost vectors all (c_1, c_2) for which $c_1x_1 + c_2x_2$ achieves its maximum over $\mathrm{conv}\{(0,0),(1,0),(0,1)\}$ at $(1,0)$.

(b) Show that the set C_j is convex (in general, not just for the example in (a)).

Remark: The set C_j is also closed under nonnegative scaling. That is, if $\mathbf{c} \in C_j$ then $\lambda\mathbf{c} \in C_j$ for all $\lambda \geq 0$. A convex set closed under nonnegative scaling is called a *convex cone*.