Math 407 – Homework 1

Due on Thursday, October 5

You are welcome to talk with other students in the class about problems but should write up solutions on your own. Solutions can be handwritten or typed but need to be legible and submitted via Gradescope by the end of the day on Thursday. You should justify all your answers in order to receive full credit.

Problem 1. Consider the the following linear program:

$$\begin{aligned} \max \quad x + y \quad \text{subject to} \quad & x \geq 0 \\ & y \geq 0 \\ & 5x + 2y \leq 10 \\ & 3x + 4y \leq 12 \\ & x - y \leq 1 \end{aligned}$$

(a) Plot the feasible region in the (x, y)-plane.
(b) Find the optimal solution and optimal value graphically.
(c) Now we replace the objective function with $x + \lambda y$. For what values of $\lambda \in \mathbb{R}$ does your optimal solution from (b) remain an optimal solution?

Problem 2. Consider a system of inequalities

$$\mathbf{a}_1^T \mathbf{x} \leq b_1, \ \ldots, \ \mathbf{a}_m^T \mathbf{x} \leq b_m$$

where $\mathbf{a}_1, \ldots, \mathbf{a}_m$ are vectors in $\mathbb{R}^n$ and $b_1, \ldots, b_m \in \mathbb{R}$. We call the jth constraint, $\mathbf{a}_j^T \mathbf{x} \leq b_j$, *redundant* if its removal does not change the set of points satisfying these inequalities, i.e.

$$\left\{ \mathbf{x} \in \mathbb{R}^n : \mathbf{a}_i^T \mathbf{x} \leq b_i \text{ for all } i \in \{1, \ldots, m\} \right\} = \left\{ \mathbf{x} \in \mathbb{R}^n : \mathbf{a}_i^T \mathbf{x} \leq b_i \text{ for all } i \in \{1, \ldots, m\} \backslash \{j\} \right\}.$$

For example, in the system of three inequalities $x_1 \leq 2,\ x_2 \leq 3,\ x_1 + x_2 \leq 6$, the third constraint $x_1 + x_2 \leq 6$ is redundant, since it is implied by the other two ($x_1 \leq 2$ and $x_2 \leq 3$), whereas neither of the first two inequalities is redundant.

Given a system of inequalities, $\mathbf{a}_1^T \mathbf{x} \leq b_1, \ldots, \mathbf{a}_m^T \mathbf{x} \leq b_m$, we'd like to know: is the jth inequality redundant? Explain how the answer depends on the optimal value of the following linear program:

$$\max \quad \mathbf{a}_j^T \mathbf{x} \quad \text{subject to} \quad \mathbf{a}_i^T \mathbf{x} \leq b_i \text{ for all } i \in \{1, \ldots, m\} \backslash \{j\}$$

Moral: It can be useful to remove redundant inequalities from a linear program before solving it in order to simplify the problem. However, in general, knowing whether or not an inequality is redundant can be as hard as solving another linear program!

Problem 3. You would like to make a shake using apples, bananas, carrots, dates, and (powdered) egg whites with a high nutritional value at a low price. The table below lists the nutritional content of these five ingredients, their price per kilogram, and the minimum amounts of vitamins/minerals/fiber/protein you would like in your shake.

Ingredients	apples	bananas	carrots	dates	egg whites	min. amt.
Vitamin A RAE (mg/kg)	0	0	8	0	0	1mg
Vitamin C (mg/kg)	46	123	60	4	0	30mg
Calcium (mg/kg)	60	50	330	390	1040	400mg
Dietary Fiber (g/kg)	24	46	28	80	0	15g
Protein (g/kg)	3	7	9	25	800	15g
Price ($/kg)	4	2	3	10	70	

based on nutritional data from `https://fdc.nal.usda.gov`

(a) Using the variables $\mathbf{x} = (a, b, c, d, e)^T$ where a denotes the kilograms of apples used, b denotes the kilograms of bananas used and so on, formulate the problem of finding the minimum cost of a shake meeting the desired nutritional requirements as a linear optimization problem.

(b) Is this linear program feasible? Why or why not?

(c) Is the feasible region of the linear program bounded? Why or why not?

(d) Is this linear program bounded? Why or why not?

Hint 1: In order to minimize (rather than maximize) a function f over a region S, we can instead maximize $-f$. That is, if $\alpha = \min\{f(\mathbf{x}) : \mathbf{x} \in S\}$, then $-\alpha = \max\{-f(\mathbf{x}) : \mathbf{x} \in S\}$.

Hint 2: There are implicit constraints on a, b, c, d, e coming from the fact that they represents amounts of physical objects!