Degree Stability of $\sum \mathbb{R}[\overline{x}]^2 + I$

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Semidefinite Programming:

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a generalization of linear programming
also solvable in polynomial time!

SDP and sums of squares:

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Why?

$$A \succeq 0 \iff A = \sum_i b_i b_i^T \iff x^T A x = \sum_i x^T b_i b_i^T x = \sum_i (b_i^T x)^2$$

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where f is k-sos mod I $\Leftrightarrow$ $f = \sum_i g_i^2$ mod I with $\deg(g_i) \leq k$.

Combinatorial problems and polynomials:

Given a graph $\mathcal{G} = (V, \mathcal{E})$, let

$$I_{\mathcal{G}} = \langle x_v^2 - x_v \ : \ v \in V \rangle + \langle x_v x_w \ : \ (v, w) \in \mathcal{E} \rangle$$

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Check out "Theta Bodies" by Gouveia, Parrilo, Thomas.

Motivation:

Many optimization problems can be expressed in terms of

$$\{f \in \mathbb{R}[x] \;:\; \deg(f) \leq d, \;\; f \geq 0 \text{ on } \mathcal{V}_{\mathbb{R}}(I)\}$$

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In practice,

$$\{f \in \mathbb{R}[x] \ : \ \deg(f) \leq d, \ \ f \text{ is k-sos mod } I\} \qquad \text{(an SDP!)}$$

where f is k-sos mod $I \Leftrightarrow f = \sum_i g_i^2$ mod I with $\deg(g_i) \leq k$.

Example: minimizing $f(x)$ over $\mathcal{V}_{\mathbb{R}}(I)$

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When does this happen?

Stability

Say that $\sum \mathbb{R}[x]^2 + I$ is **stable** if there exists $d : \mathbb{N} \to \mathbb{N}$ so that for every f that is sos mod I, f is $d(\deg(f))$-sos mod I.

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Main Question: When is $\sum \mathbb{R}[x]^2 + I$ stable ?

Theta Bodies

For theta body enthusiasts . . .

$\sum \mathbb{R}[x]^2 + I$ stable

$\Rightarrow \{f \text{ linear, } N\text{-sos mod } I\} = \{f \text{ linear, } (N+1)\text{-sos mod } I\} = \dots$

$\Rightarrow TH_N(I) = TH_{N+1}(I) = \dots$

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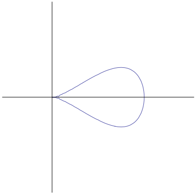
Is $\sum \mathbb{R}[x]^2 + I$ stable ? No for $\mathcal{V}_{\mathbb{R}}(I)$ compact with $\dim \geq 2$.



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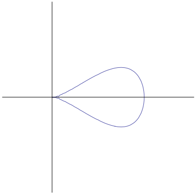

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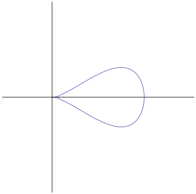

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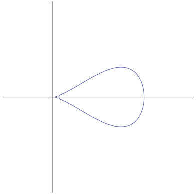

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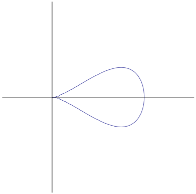
suppose $\quad x = \sum (g_i)^2 + h(y^2 - x^3 + x^4) \ldots$



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$$x \notin \sum \mathbb{R}[x,y]^2 + \langle y^2 - x^3 + x^4 \rangle.$$

$\Rightarrow$ we need *unbounded* degree to represent $x + c$ as $c \to 0$

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For $w \in \mathbb{R}^n$, define ...

- $\deg_w(\sum_\alpha a_\alpha x^\alpha) := max\{w^T\alpha \ : \ a_{\alpha\neq 0}\}$.
- $in_w(f) :=$ sum of terms with maximal weight ($\deg_w$)

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$\Rightarrow$ we can reduce g_i mod I

Example: $I = \langle y^2 - x^3 - x \rangle$ $\quad in_{(2,3)(I)} = \langle y^2 - x^3 \rangle$ real radical

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Both have $\deg_{(2,3)} = 16$.

Theorem

If $w \in \mathbb{R}^n_{>0}$ s.t. $in_w(I)$ is real radical, then for any $f \in \sum \mathbb{R}[x]^2 + I$, there are g_i with $2\deg_w(g_i) \leq \deg_w(f)$ and $f \equiv \sum_i (g_i)^2 \mod I$.

Corollary:

If there exists $w \in \mathbb{R}^n_{>0}$ so that $in_w(I)$ is real radical, then $\sum \mathbb{R}[x]^2 + I$ is stable.

Sketch of the proof . . .

- Start with $f = \sum_i g_i^2 \bmod I$.

Sketch of the proof . . .

- Start with $f = \sum_i g_i^2$ mod I.

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Questions:

- How can we check this condition $(\exists w \in (\mathbb{R}_{>0})^n \ldots)$?

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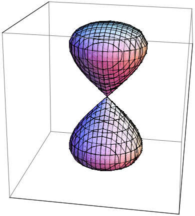

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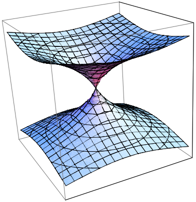

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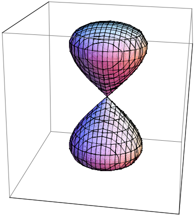

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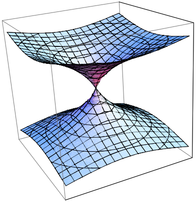

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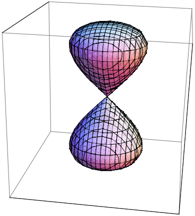

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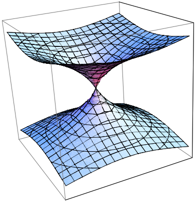

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Thanks!