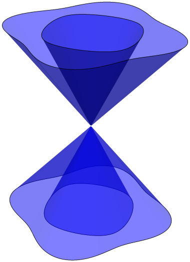
Hyperbolic Polynomials, Interlacers, and Sums of Squares

Cynthia Vinzant

University of Michigan



joint work with Mario Kummer and Daniel Plaumann

Hyperbolic Polynomials

A homogeneous polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]_d$ is *hyperbolic* with respect to a point $e \in \mathbb{R}^n$ if $f(e) \neq 0$ and for every $x \in \mathbb{R}^n$, all roots of $f(te + x) \in \mathbb{R}[t]$ are real.

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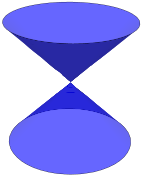

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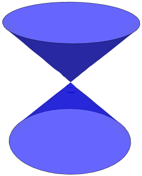

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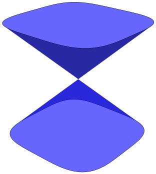

$x_1^4 - x_2^4 - x_3^4$

not hyperbolic

Hyperbolicity Cones

Its *hyperbolicity cone*, denoted $C(f, e)$,
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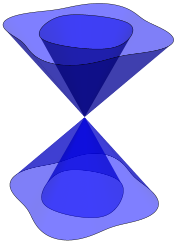

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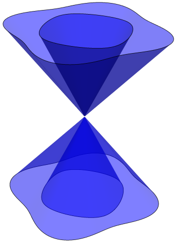

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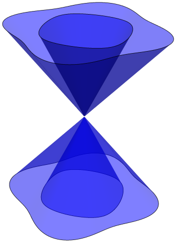

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- f is hyperbolic with respect to any point $a \in C(f, e)$.

One can use interior point methods to optimize a linear function over an affine section of a hyperbolicity cone, Güler (1997), Renegar (2006). This solves a *hyperbolic program*.

Two Important Examples of Hyperbolic Programming

f		
e		
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	Linear Programming	
f	$\prod_i x_i$	
e	$(1, \ldots, 1)$	
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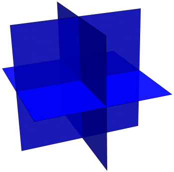
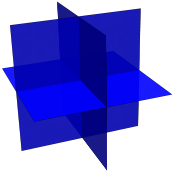
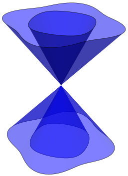
		

Two Important Examples of Hyperbolic Programming

	Linear Programming	Semidefinite Programming
f	$\prod_i x_i$	$\det \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{1n} & \cdots & x_{nn} \end{pmatrix}$
e	$(1, \ldots, 1)$	Id_n
$C(f, e)$	$(\mathbb{R}_+)^n$	positive definite matrices
		

Connections to Multiaffine Polynomials and Matroids

A polynomial f is multiaffine if it has degree one in each variable.

Example: $f = x_1x_2 + x_1x_3 + x_2x_3$

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Theorem (Choe, Oxley, Sokal, Wagner (2004))

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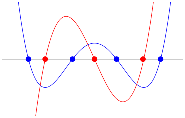
If f is multiaffine and real stable then the monomials in the support of f form the bases of a matroid.

For any representable matroid there is a multiaffine real stable polynomial whose support is the collection of its bases.

Example: $f = x_1x_2 + x_1x_3 + x_2x_3 \quad \longrightarrow \quad \{\{1,2\},\{1,3\},\{2,3\}\}$

Interlacing Derivatives

If all roots of $p(t)$ are real, then the roots of $p'(t)$ are real and interlace the roots of $p(t)$.



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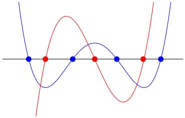
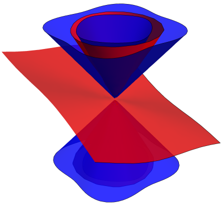
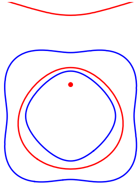


For any direction $a \in C(f, e)$ the polynomial

$$D_a(f) = \sum_i a_i \frac{\partial f}{\partial x_i} = \left(\frac{\partial}{\partial t} f(ta + x) \right) \Big|_{t=0}$$

is hyperbolic and *interlaces* f.





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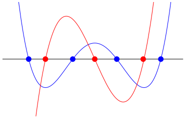

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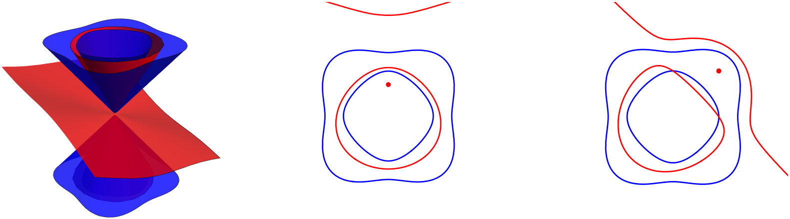
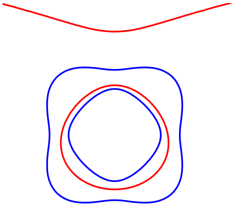
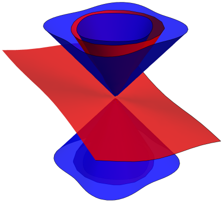
is hyperbolic and *interlaces* f. (Not true for $a \notin C(f, e)$).



The Convex Cone of Interlacers

$$Int(f, e) = \{g \in \mathbb{R}[x_1, \ldots, x_n]_{d-1} \ : \ g(e) > 0 \text{ and } g \text{ interlaces } f\}$$





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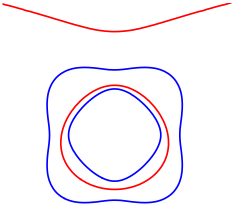

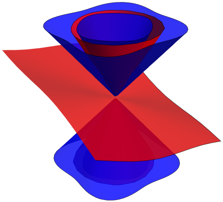

Theorem

If f is square free and hyperbolic w.r.t. $e \in \mathbb{R}^n$, then

$$Int(f, e) = \{g \ : \ D_e f \cdot g - f \cdot D_e g \geq 0 \ \text{ on } \ \mathbb{R}^n\}.$$

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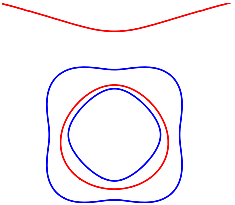

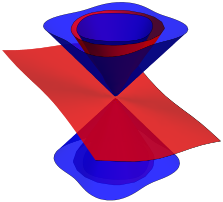

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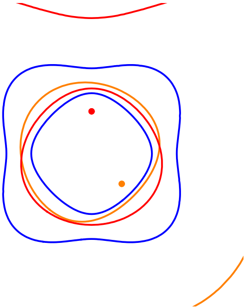
This is a convex cone in $\mathbb{R}[x_1, \ldots, x_n]_{d-1}$.

Special Interlacers $g = D_a f$

Theorem

If $f \in \mathbb{R}[x_1, \ldots, x_n]_d$ is square-free and hyperbolic w.r.t $e \in \mathbb{R}^n$,

$$\overline{C(f,e)} = \{\ a \in \mathbb{R}^n\ :\ D_e f \cdot D_a f - f \cdot D_e D_a f\ \geq\ 0 \quad on \quad \mathbb{R}^n\ \}.$$



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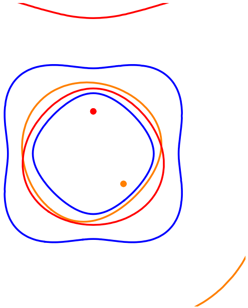
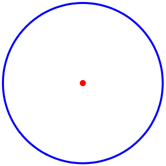


This writes the hyperbolicity cone $\overline{C(f,e)}$ as a slice of the cone of *nonnegative polynomials*.

Example: the Lorentz cone

$f(x) = x_1^2 - x_2^2 - \ldots - x_n^2$ $\quad e = (1, 0, \ldots, 0)$



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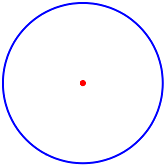

$D_e f \cdot D_a f - f \cdot D_e D_a f$

$$= \ (2x_1)(2a_1x_1 - \sum_{j\neq 1} 2a_jx_j) - (x_1^2 - \sum_{j\neq 1} x_j^2)(2a_1)$$

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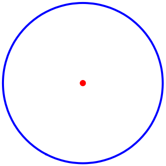

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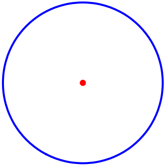

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$$\Rightarrow \quad \overline{C(f,e)} = \left\{ a \in \mathbb{R}^n : \begin{pmatrix} a_1 & -a_2 & \ldots & -a_n \\ -a_2 & a_1 & & 0 \\ \vdots & & \ddots & \vdots \\ -a_n & 0 & \ldots & a_1 \end{pmatrix} \succeq 0 \right\}$$

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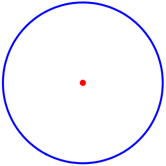

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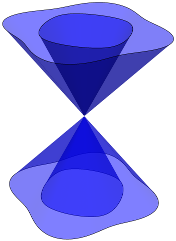
(determinant $= a_1^{n-2} f(a)$)

Sums of Squares Relaxation

Corollary

$$\{a \in \mathbb{R}^n \ : \ D_e f \cdot D_a f - f \cdot D_e D_a f \ \text{ is a sum of squares } \} \subseteq \overline{C(f, e)}.$$



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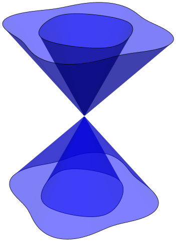
$\{a \in \mathbb{R}^n \ : \ D_e f \cdot D_a f - f \cdot D_e D_a f$ *is a sum of squares* $\} \subseteq \overline{C(f, e)}$.

the projection of a spectrahedron!



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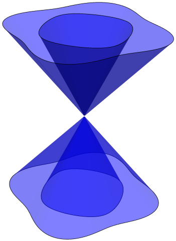
Theorem

If some power of f has a determinantal representation $f^r = \det(\sum_i x_i M_i)$ where $M_1, \ldots, M_n$ are real symmetric matrices and $\sum_i e_i M_i \succ 0$, then this relaxation is exact.



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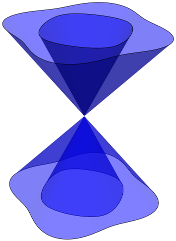

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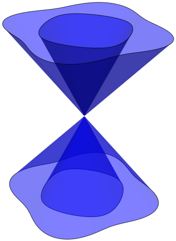
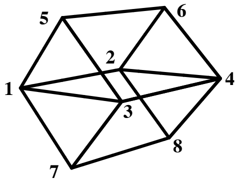
Answer: No.



A Counterexample: The Vámos Matroid

$$f(x_1, \ldots, x_8) = \sum_{I \subset \binom{[8]}{4} \setminus C} \prod_{i \in I} x_i,$$



$$C = \{\{1,2,3,4\}, \{1,2,5,6\}, \{1,2,7,8\}, \{3,4,5,6\}, \{3,4,7,8\}\}.$$

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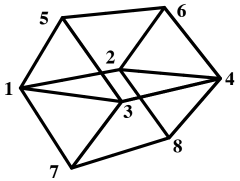

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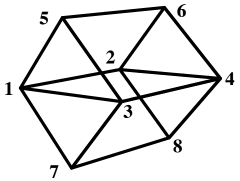
Wagner, Wei (2009): f is hyperbolic w.r.t. $(\mathbb{R}_+)^n$ *(i.e. real stable)*

A Counterexample: The Vámos Matroid

$$f(x_1, \ldots, x_8) = \sum_{I \subset \binom{[8]}{4} \setminus C} \prod_{i \in I} x_i,$$



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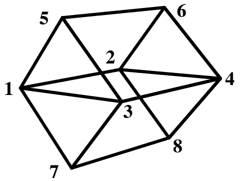

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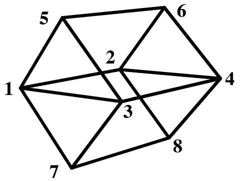

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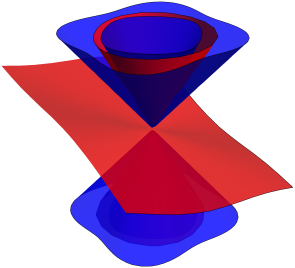
Corollary. Brändén's result.

Last Thoughts



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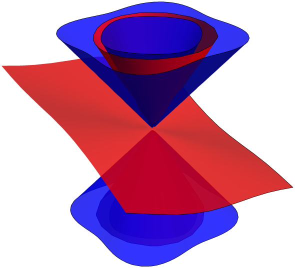

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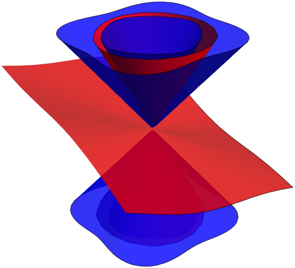

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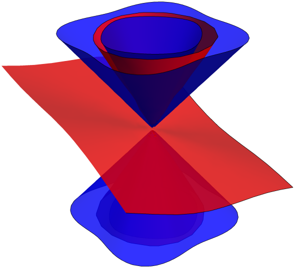

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