Traditionally in ergodic theory we deal with a measure-preserving map of a space of total measure one. This allows for a probability interpretation, with subsets being “events” of probability equal to their measure. The Birkhoff Ergodic Theorem says that if the map is measure-theoretically indecomposable (“ergodic”) then the density of return-times of a.e. point to the set is equal to its measure: that “time average equals space average”. But infinite measures can occur naturally as well. Here the times of return to a set of finite measure must have density zero; yet in some cases it can exhibit nice additional structure: a “Hausdorff dimension” between 0 and 1, and a Hausdorff measure of that dimension. This last leads to a new form of the statement that time average equals space average (an order-two ergodic theorem).

We will survey some examples coming from probability theory (renewal processes) and from dynamics (adic transformations). Interesting example of this last come from return maps to measure-zero Cantor subsets of a circle rotation.

Joint work with Marina Talet.