Solutions to the first midterm

Math 134

Problem 1. Let $r$ be a rational number, $r > 1$. Let $f$ be defined by

$$f(x) = (1 + x)^r - (1 + r x) \text{ for } x \geq -1.$$ 

1.1 Find the critical points of $f$. Determine the intervals where $f$ is increasing and where $f$ is decreasing.

Solution: $f$ is differentiable on $(-1, +\infty)$ and

$$f'(x) = r(1 + x)^{r-1} - r \text{ for } x > -1.$$ 

If $f'(x) = 0$ then $(1 + x)^{r-1} - 1 = 0$ which implies that $x = 0$ is a critical point. For $-1 < x < 0$, $0 < (1 + x)^{r-1} < 1$, therefore $f'(x) < 0$, and $f$ is decreasing in $(-1, 0)$. For $x > 0$, $(1 + x)^{r-1} > 1$, thus $f'(x) > 0$ and $f$ is increasing on $(0, +\infty)$. Hence $x = 0$ is a minimum of $f$.

1.2 Determine the concavity of the graph of $f$.

Solution: $f'$ is differentiable in $(-1, +\infty)$ and

$$f''(x) = r(r - 1)(1 + x)^{r-2} \text{ for } x > -1.$$ 

For $x > -1$, $f''(x) > 0$ and $f$ is concave up on $(-1, +\infty)$.

1.3 Determine the value of $f$ at $-1$. Determine the behavior of $f(x)$ as $x$ approaches $+\infty$.

Solution: $f(-1) = r - 1$. In order to determine the behavior of $f(x)$ as $x$ approaches $+\infty$ note that for $x > 0$

$$f(x) = x^r \left(1 + \frac{1}{x}\right)^r - x \left(r + \frac{1}{x}\right) = x^r \left[\left(1 + \frac{1}{x}\right)^r - \frac{1}{x^{r-1}} \left(r + \frac{1}{x}\right)\right].$$

Note that

$$\left(1 + \frac{1}{x}\right)^r \rightarrow 1 \text{ as } x \rightarrow +\infty,$$

and since $r - 1 > 0$, then

$$\frac{1}{x^{r-1}} \left(r + \frac{1}{x}\right) \rightarrow 0 \text{ as } x \rightarrow +\infty.$$
Thus
\[ x^r \left[ \left( 1 + \frac{1}{x} \right)^r - \frac{1}{x^{r-1}} \left( r + \frac{1}{x} \right) \right] \to +\infty \text{ as } x \to +\infty. \]

**Problem 2.** Let \( f : [0, 1] \to \mathbb{R} \). Suppose that the function \( f \) has the following property
\[ |f(x) - f(y)| \leq (x - y)^2 \text{ for all } x, y \in [0, 1]. \]

2.1 Show that the function \( f \) is continuous on \([0,1]\).

**Solution:** Option1: Let \( c \in [0,1] \). Given \( \epsilon > 0 \), let \( \delta = \sqrt{\frac{\epsilon}{2}} \), then for \( y \in [0,1] \) such that \( |c - y| < \delta \) then
\[ |f(c) - f(y)| \leq (c - y)^2 < \delta^2 = \frac{\epsilon}{2} < \epsilon. \]
This proves that \( f \) is continuous at each point \( c \in [0,1] \).

Option2: Let \( c \in (0,1) \), then by hypothesis
\[ -(c - y)^2 \leq f(c) - f(y) \leq (c - y)^2. \]
Hence
\[ f(c) - (c - y)^2 \leq f(y) \leq f(c) + (c - y)^2. \]
Note that
\[ \lim_{y \to c} (f(c) - (c - y)^2) = \lim_{y \to c} (f(c) + (c - y)^2) = f(c). \]
Therefore by the pinching theorem
\[ \lim_{y \to c} f(y) = f(c). \]
This implies that \( f \) is continuous in \((0,1)\). If \( c = 0 \) then for \( y \in [0,1] \) we have
\[ f(0) - y^2 \leq f(y) \leq f(0) + y^2. \]
Note that
\[ \lim_{y \to 0^+} f(0) - y^2 = \lim_{y \to 0^+} f(0) + y^2 = f(0). \]
Therefore by the pinching theorem
\[ \lim_{y \to 0^+} f(y) = f(0). \]
This implies that $f$ continuous to the right at 0. A similar proof shows that $f$ is continuous to the left at 1. Hence $f$ is continuous on $[0,1]$.

Option 3: Do problem 2.2 first then argue that every differentiable function is continuous. This would prove continuity on the open interval $(0,1)$. The continuity at the end points can be dealt with by one of the arguments above.

2.2 Show that the function $f$ is differentiable in $(0,1)$, and that $f'(x) = 0$ for every $x \in (0,1)$.

**Solution:** Let $x \in (0,1)$ consider the ratio $\frac{f(x+h)-f(x)}{h}$ for $h$ small enough. The hypothesis above ensures that for $|h| > 0$ and small enough

$$\left| \frac{f(x+h) - f(x)}{h} \right| \leq \frac{h^2}{|h|},$$

which implies that

$$0 \leq \left| \frac{f(x+h) - f(x)}{h} \right| \leq |h|.$$

Since $\lim_{h \to 0} |h| = 0$, the pinching theorem implies that

$$\lim_{h \to 0} \left| \frac{f(x+h) - f(x)}{h} \right| = 0.$$

Thus

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 0. \quad \blacksquare$$

**Problem 3.**

3.2 Let $f$ be a differentiable function on $(0, +\infty)$. Assume that $f'(x) \to 0$ as $x \to +\infty$. Let $g(x) = f(x+1) - f(x)$ show that $g(x) \to 0$ as $x \to +\infty$.

**Solution:** Since $f$ is differentiable in any interval of the form $[x, x+1]$ for $x > 0$, and $f$ is differentiable in the open interval $(x, x+1)$, the mean value theorem ensures that for each such $x$ there is $c_x \in (x, x+1)$ so that

$$g(x) = f(x+1) - f(x) = f'(c_x)(x + 1 - x) = f'(c_x).$$

Note that as $x \to +\infty$, since $c_x > x$ then $c_x \to +\infty$. Hence as $x \to +\infty$, $f'(c_x) \to 0$ (by hypothesis) and so $g(x) \to 0. \quad \blacksquare$