Summary

$$x^2 f''(x) + x f'(x) + (\mu^2 x^2 - \nu^2) f(x) = 0 \qquad (\star)$$

Change of variable $g(x) = f(x/\mu)$ leads to the Bessel equation of order ν

$$x^2 g''(x) + x g'(x) + (x^2 - \nu^2) g(x) = 0 \qquad (\star\star)_\nu$$

- if $\nu \notin \mathbb{N}$

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}$$

$$J_{-\nu}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k-\nu+1)} \left(\frac{x}{2}\right)^{2k-\nu}$$

2 linearly independent solutions to $(\star\star)$

Recall $\Gamma(k+\nu+1) = (k+\nu)(k+\nu-1)\ldots 1$

If $\nu \in \mathbb{N}$ $\nu = n$ $J_{-n}(x) = (-1)^n J_n(x)$, J_{-n} & J_n not linearly independent. Here

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!} \left(\frac{x}{2}\right)^{2k+n}$$

To obtain a solution to

$$x^2 g''(x) + x g'(x) + (x^2 - n^2) g(x) = 0 \qquad (\star\star)_n$$

which is linearly independent from J_n we consider Bessel functions of 2nd kind for $\nu \notin \mathbb{N}$

$$Y_\nu(x) = \frac{\cos(\nu\pi) J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}$$ which a

solution to $(\star\star)_\nu$. Define $Y_n(x) = \lim_{\nu \to n} Y_\nu(x)$

J_n and Y_n linearly independent solutions to $(\star\star)$

Remark: J_ν and Y_ν are linearly indep. solutions to Bessel's equations of order ν.

Orthogonal sets of Bessel functions

Recall

$$x^2 f''(x) + x f'(x) + (\mu^2 x^2 - \nu^2) f(x) = 0 \qquad (\star)$$

which we rewrite as

$$x f''(x) + f'(x) - \frac{\nu^2}{x} f(x) + \mu^2 x f(x) = 0$$

$$[x f'(x)]' - \frac{\nu^2}{x} f(x) + \mu^2 x f(x) = 0$$

Sturm-Liouville equation $(r f')' + p f + \mu^2 w f = 0$

$$r(x) = x \qquad p(x) = -\frac{\nu^2}{x} \qquad w(x) = x$$

This equation on $[a,b]$, $0<a<b<\infty$ and with boundary conditions

$$\alpha f(a) + f'(a) = 0 \qquad \beta f(b) + f'(b) = 0 \qquad (**)$$

is a regular Sturm-Liouville problem. The eigenfunctions are of the form

$$f_{\mu,\nu}(x) = c_\mu J_\nu(\mu x) + d_\mu Y_\nu(\mu x)$$

where μ, c_μ, d_μ are chosen so that $(**)$ holds. We obtain an orthonormal basis for $L^2_w(a,b)$ with functions of the form $f_{\mu,\nu}$ above

We want to pass on

$$x f''(x) + f'(x) - \frac{\nu^2}{x} f(x) + \mu^2 x f(x) = 0$$

on $[0,b]$ with $\nu \geq 0$.

Q: What happens to $Y_\nu(\mu x)$ near 0?

$$Y_\nu(\mu x) = \frac{\cos(\nu\pi)\, J_\nu(\mu x) - J_{-\nu}(\mu x)}{\sin(\nu\pi)} \sim \frac{-1}{\Gamma(1-\nu)} \left(\frac{\mu x}{2}\right)^{-\nu} \qquad \nu \notin \mathbb{N}$$

as $x \to 0$

$$Y_n(\mu x) \sim -\frac{(n-1)!}{\pi} \left(\frac{\mu x}{2}\right)^{-n} \qquad x \to 0 \qquad n = 1, 2, \dots$$

$$Y_0(\mu x) \sim \frac{2}{\pi} \log \frac{\mu x}{2} \qquad x \to 0 \qquad \left(\lim_{x \to 0} |Y_\nu(x)| = +\infty\right)$$

Consider for $\nu \geq 0$

$$(SL) \begin{cases} x f''(x) + f'(x) - \frac{\nu^2}{x} f(x) + \mu^2 x f(x) = 0 \\ f(0+) \text{ exists and finite} \qquad \beta f(b) + \beta' f'(b) = 0 \end{cases}$$

(boundary condition at 0)

$\Downarrow$

$d\mu = 0$ solutions of the form $f_{\mu,\nu}(x) = J_\nu(\mu x)$

"**Proposition**: The (SL) problem satisfies:

(1) all eigenvalues are real

(2) Eigenfunctions corresponding to distinct eigenvalues are orthogonal in $L^2_w(0,b)$.

(3) The eigenspaces are 1-dimensional"

The solutions are of the form $f(x) = J_\nu(\mu x)$, we now adjust the boundary values. $f'(x) = \mu J_\nu'(\mu x)$

$$\beta J_\nu(\mu b) + \beta' \mu J_\nu'(\mu b) = 0 \quad \text{boundary cond.}$$

Let $\lambda = \mu b$. Case 1: $\beta' = 0$ $(\beta \neq 0)$ $\boxed{J_\nu(\lambda) = 0}$

Case 2: $\beta' \neq 0$ $\ell = b\beta/\beta'$ & (multiplying by $\frac{b}{\beta'}$) we have

$$\boxed{\ell J_\nu(\lambda) + \lambda J_\nu'(\lambda) = 0}$$

§5.3 in both cases there is an infinite sequence $\{\lambda_k\}_{k=1}^{\infty}$ of positive solutions, the eigenvalues $\boxed{\mu_k^2 = \lambda_k^2/b^2}$

Lemma: 0 is an eigenvalue of (SL) iff $\beta/\beta' = -\nu/b$, in which case the eigenfunction is $f(x) = x^\nu$ - If $\beta' = 0$ or if $\beta/\beta' \geq -\nu/b$ there are no negative eigenvalues

Remarks: ① $\mu = 0$ $\quad x^2 f''(x) + x f'(x) - \nu^2 f(x) = 0$ $\quad$ § 4.3

$f(x) = c_1 x^\nu + c_2 x^{-\nu}$ $\quad \nu > 0$ $\quad$ or $\quad f(x) = c_1 + c_2 \log x$ $\quad \nu = 0$

since f continuous at 0 $\quad c_2 = 0$ in both cases - The boundary condition $\beta f(b) + \beta' f'(b) = 0$ becomes

$\nu = 0$ $\quad \beta c_1 = 0$ $\quad \beta = 0$ $\quad \beta/\beta' = -\nu/b$

$\nu \neq 0$ $\quad \beta(c_1 b^\nu + c_2 b^{-\nu}) + \beta'(c_1 \nu b^{\nu-1} - c_2 \nu b^{-\nu-1}) = 0$

$$c_1 b^{\nu-1}(\beta b + \beta' \nu) + c_2 b^{-\nu-1}(\beta b + \beta' \nu) = 0$$

$$(\beta b + \beta' \nu)\left(c_1 b^\nu + c_2 b^{-\nu}\right)\frac{1}{b} = 0 \quad \text{iff} \quad \beta b + \beta' \nu = 0$$

② To investigate the negative eigenvalues set $\mu = ik$ ...

Summary : We have constructed a complete family of orthogonal functions on $[0,b]$ w.r.t the weight $w(x) = x$ of the form

$$f_k(x) = J_\nu\left(\frac{\lambda_k x}{b}\right)$$

To make them useful we need to identify the norms to have an orthonormal basis

$$\|f_k\|_w^2 = \int_0^b |f_k|^2 x\,dx = \int_0^b \left(J_\nu\left(\frac{\lambda_k x}{b}\right)\right)^2 x\,dx$$

Let $\mu > 0$, $b > 0$ & $\nu \geq 0$

$$(SL) \begin{cases} x f''(x) + f'(x) - \frac{\nu^2}{x} f(x) + \mu^2 x f(x) = 0 \\ f(0+) \text{ exists and finite} \qquad \beta f(b) + \beta' f'(b) = 0 \end{cases}$$

then $$\int_0^b J_\nu(\mu x)^2 x\,dx = \frac{b^2}{2} J_\nu'(\mu b)^2 + \frac{\mu^2 b^2 - \nu^2}{2\mu^2} J_\nu(\mu b)^2$$

let $\lambda = \mu b$ if $\beta' = 0$ $\quad J_\nu(\lambda) = 0$ and

$$\int_0^b J_\nu\left(\frac{\lambda}{b} x\right)^2 x\,dx = \frac{b^2}{2} J_\nu'(\lambda)^2 = \frac{b^2}{2} J_{\nu+1}(\lambda)^2$$

because $J_\nu'(x) = -J_{\nu+1}(x)$

if $c J_\nu(\lambda) + \lambda J_\nu'(\lambda) = 0$

$$\int_0^b J_\nu\left(\frac{\lambda}{b} x\right)^2 dx = \frac{b^2(\lambda^2 - \nu^2 + c^2)}{2\lambda^2} J_\nu(\lambda)^2$$

Theorem: Suppose $\nu \geq 0$ $b > 0$ $w(x) = x$

a) Let $\{\lambda_k\}_{k=1}^{\infty}$ be the positive zeros of $\bar{J}_\nu(x)$ and let

$\phi_k(x) = \bar{J}_\nu\left(\frac{\lambda_k x}{b}\right)$ then $\{\phi_k\}_{k=1}^{\infty}$ orthogonal basis for $L^2_w(0,b)$

$$\|\phi_k\|_w^2 = \frac{b^2}{2}\,\bar{J}_{\nu+1}^2(\lambda_k)$$

b) If $c \geq -\nu$ - let $\{\tilde{\lambda}_k\}$ be the positive zeros of $c\bar{J}_\nu(x) + x\bar{J}_\nu'(x)$

and $\psi_k = \bar{J}_\nu\left(\frac{\tilde{\lambda}_k x}{b}\right)$. If $c > -\nu$ $\{\psi_k\}_{k=1}^{\infty}$ orthogonal bases

for $L^2_w(0,b)$ - If $c = -\nu$ $\{\psi_k\}_{k=0}^{\infty}$ orthogonal basis for

$L^2_w(0,b)$ where $\psi_0(x) = x^\nu$.

$$\|\psi_k\|_w^2 = \frac{b^2(\tilde{\lambda}_k^2 - \nu^2 + c^2)}{2\tilde{\lambda}_k^2}\,\bar{J}_\nu(\tilde{\lambda}_k)^2 \qquad \|\psi_0\|_w^2 = \frac{b^{2\nu+2}}{2\nu+2}$$

$k \geq 1$

Solve the wave equation in polar coordinates $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$

(1) $u_{tt} = c^2 (u_{rr} + r^{-1} u_r + r^{-2} u_{\theta\theta})$ in the disc of radius b, Θ 2π-periodic and $\nu = n \in \mathbb{N}$

$$\begin{cases} \Theta''(\theta) + \nu^2 \Theta(\theta) = 0 \Rightarrow \Theta(\theta) = a_\nu \cos\nu\theta + b_\nu \sin\nu\theta \\ r^2 R''(r) + r R'(r) + (\mu^2 r^2 - \nu^2) R(r) = 0 \quad (\star) \\ T''(t) + c^2\mu^2 T(t) = 0 \Rightarrow T(t) = \alpha_{c\mu} \cos c\mu t + \beta_{c\mu} \sin c\mu t \end{cases}$$

$(\star)$ becomes $r^2 R''(r) + r R' + (\mu^2 r^2 - n^2) R(r) = 0$

Assume we want a solution u continuous in the whole disc and such that $u(b,\theta,t) = 0$ then R cannot blow up at 0 and $R(b) = 0$. Hence since $\lambda = \mu b$

$$R(r) = J_n(\mu r) = J_n\left(\frac{\lambda}{b} r\right)$$

The problem of finding the solution with given initial conditions reduces to expanding the initial data in a series

involving Θ and R as defined above

$$\underbrace{\sum_{n,k} \left(c_{nk} \cos n\theta + d_{nk} \sin n\theta\right) J_n\left(\frac{\lambda_{kn} r}{b}\right)}_{f(r)}$$

$$u(r,\theta,t) = \sum_{n,k} \left(c_{nk} \cos n\theta + d_{nk} \sin n\theta\right) J_n\left(\frac{\lambda_{kn} r}{b}\right) \left(a_{nk} \cos \frac{c\lambda_{kn} t}{b} + b_{nk} \sin \frac{c\lambda_{kn} t}{b}\right)$$

Assume $u_{tt} = c^2 (u_{rr} + r^{-1} u_r + r^{-2} u_{\theta\theta})$ in D_b

$u(b,\theta,t) = 0 \Rightarrow R(b) = 0 \quad (\beta' = 0)$

$u(r,\theta,0) = f(r)$ and $u_t(r,\theta,0) = 0$

$$f(r) = \sum_k \alpha_k J_0\left(\frac{\lambda_{k0} r}{b}\right) \qquad \alpha_k = c_{0k}$$

$$u(r,\theta,0) = \sum_{n,k} \left(c_{nk} \cos n\theta + d_{nk} \sin n\theta\right) a_{nk} J_n\left(\frac{\lambda_{kn} r}{b}\right)$$

$$u_t(r,\theta,0) = \sum_{n,k} \left(c_{nk} \cos n\theta + d_{nk} \sin n\theta\right) b_{nk} c\frac{\lambda_{kn}}{b} J_n\left(\frac{\lambda_{kn} r}{b}\right) = 0$$

$\Rightarrow b_{nk} = 0$

$$f(r) = \sum_k \alpha_k J_0\left(\frac{\lambda_{k0} r}{b}\right)$$

$$\|$$

$$u(r,\theta,0) = \sum_{n,k} (c_{nk}\cos n\theta + d_{nk}\sin n\theta)\, a_{nk} J_n\left(\frac{\lambda_{kn} r}{b}\right)$$

$d_{nk} = 0 \qquad c_{nk} = 0 \quad$ for $\quad n > 0 \quad$ / or $a_{nk} = 0$

$$u(r,\theta,0) = \sum c_{0k} a_{0k} J_0\left(\frac{\lambda_{k0} r}{b}\right)$$

$$u_t(r,\theta,0) = \sum_{n\,k} (c_{nk}\cos n\theta + d_{nk}\sin n\theta)\, b_{nk} \frac{c\lambda_k^n}{b} J_n\left(\frac{\lambda_k^n r}{b}\right) = 0$$

$$= \sum c_{0k}\, b_{0k}\, \frac{c\lambda_k^n}{b} J_0\Big(\underbrace{\frac{\lambda_k^n r}{b}}_{\neq 0}\Big) = 0 \qquad \lambda_k^n \neq 0$$

$$\Rightarrow b_{0k} = 0$$

$$u(r,\theta,t) = \sum_k \underbrace{c_{0k}\, a_{0k}}_{\alpha_k} J_0\left(\frac{\lambda_{k0} r}{b}\right) \cos \frac{c\lambda_{k0}}{b} t$$

$$\lambda_{k0} = \lambda_k$$

Final exam : Tuesday June 9 : 2:30 – 4:20 (Optional)

If you decide not to take the exam, your work up to now including homeworks 1–6, participation and midterm will be counted for 100% of your grade.

If you decide to take the exam it will count for 30% of your grade. Your final grade will be the best of the following two numbers:

1. your grade including the final exam
2. your grade computed as if you had opted not to take the final exam.