§ 4.4 problem 7 Solve the following Dirichlet problem

$$\Delta u = 0 \quad \text{in} \quad S = \{(r,\theta) : 0 < r_0 \le r \le 1 \,,\ 0 \le \theta \le \beta\}$$

$$u(r_0,\theta) = u(1,\theta) = 0 \qquad u(r,0) = g(r) \qquad u(r,\beta) = h(r)$$

(cf. problem 10 § 3.5)

$$u(r,\theta) = R(r)\,\Theta(\theta)$$

$$\begin{cases} \Theta''(\theta) + \nu^2 \Theta(\theta) = 0 \\ r^2 R''(r) + r R'(r) + \nu^2 R(r) = 0 \end{cases}$$

$$u(r_0,\theta) = R(r_0)\,\Theta(\theta) = 0$$
$$u(1,\theta) = R(1)\,\Theta(\theta) = 0$$

$$\rightsquigarrow \quad R(r_0) = R(1) = 0$$

Thus we need to solve

$$\begin{cases} r^2 R''(r) + r R'(r) + \nu^2 R(r) = 0 \\ R(r_0) = R(1) = 0 \end{cases}$$

$$\begin{cases} r^2 R''(r) + r R'(r) + \nu^2 R(r) = 0 \\ R(r_0) = R(1) = 0 \end{cases}$$

Change of variable $t = \ln r$ define $T(t) = R(e^t)$

compute the equation T satisfies solve the appropriate boundary data (transform $R(r_0) = R(1) = 0$ onto conditions

$$\begin{cases} T(0) = R(1) = 0 \\ T(\ln r_0) = R(r_0) = 0 \end{cases}$$

$$\begin{cases} r^2 R''(r) + r R'(r) + \nu^2 R(r) = 0 \\ R(r_0) = R(1) = 0 \end{cases}$$

Change of variable $t = \ln r$ define $T(t) = R(e^t)$

$$\frac{dT}{dt} = e^t R'(e^t) \qquad \frac{d^2T}{dt^2} = e^{2t} R''(e^t) + e^t R'(e^t)$$

$$T_{tt} - \nu^2 T(t) = e^{2t} R''(e^t) + e^t R'(t) + \nu^2 R(e^t) = 0$$

with $T(0) = 0$ $\quad T(\ln r_0) = 0$ (recall $\log r_0 < 0$)

$$T(t) = \sin \nu t \qquad T(\ln r_0) = \sin \nu \ln r_0 = 0$$

$$\nu \ln r_0 = n\pi$$

$$\boxed{\nu = \frac{n\pi}{\ln r_0}}$$

$$T_n(t) = \sin\left(\frac{n\pi t}{\ln r_0}\right)$$

$$\boxed{R(r) = \sin\left(\frac{n\pi \ln r}{\ln r_0}\right)}$$

complete orthogonal family

$$\textcircled{H}''(\theta) - \left(\frac{n\pi}{\ln r_0}\right)^2 \textcircled{H}(\theta) = 0$$

$$\textcircled{H}(\theta) = a_n e^{n\pi\theta/\ln r_0} + b_n e^{-n\pi\theta/\ln r_0}$$

$$u(r,\theta) = \sum_{n=1}^{\infty} \sin\frac{n\pi \ln r}{\ln r_0}\left(a_n e^{n\pi\theta/\ln r_0} + b_n e^{-n\pi\theta/\ln r_0}\right)$$

$$u(r,0) = \sum_{n=1}^{\infty} \sin\frac{n\pi \ln r}{\ln r_0}\left(a_n + b_n\right) = g(r)$$

$$u(r,\beta) = \sum_{n=1}^{\infty} \sin\frac{n\pi \ln r}{\ln r_0}\left(a_n e^{n\pi\beta/\ln r_0} + b_n e^{-n\pi\beta/\ln r_0}\right) = h(r)$$

Q: How do we find a_n & b_n in terms of $g(r)$ & $h(r)$?

$\left\{ \sin \frac{n\pi t}{\ln r_0} \right\}_{n=1}^{\infty}$ complete orthogonal family in $L^2(\ln r_0, 0)$

then $\left\{ \sin \frac{n\pi \ln r}{\ln r_0} \right\}_{n=1}^{\infty}$ is

$\left\{ \sin \frac{n\pi t}{\ln r_0} \right\}_{n=1}^{\infty}$ complete orthogonal family in $L^2(\ln r_0, 0)$ then

$\left\{ \sin \frac{n\pi \ln r}{\ln r_0} \right\}_{n=1}$ is a complete orthogonal family in $L^2_w(r_0, 1)$, $w(r) = \frac{1}{r}$

- $\int_{r_0}^{1} \sin \frac{n\pi \ln r}{\ln r_0} \cdot \sin \frac{m\pi \ln r}{\ln r_0} \frac{dr}{r} = \int_{\ln r_0}^{0} \sin \frac{n\pi u}{\ln r_0} \sin \frac{m\pi u}{\ln r_0} du =$

$\left(\begin{array}{l} u = \ln r \\ du = \frac{dr}{r} \end{array} \right)$

$= \begin{cases} 0 & \text{if } n \neq m \\ -\frac{\ln r_0}{2} & \text{if } n = m \end{cases}$

- $\int_{r_0}^{1} f(r) \sin \frac{n\pi \ln r}{\ln r_0} \frac{dr}{r} = \int_{\ln r_0}^{0} f(e^u) \sin \frac{n\pi u}{\ln r_0} du$

If $\int_{r_0}^{1} f(r) \sin \frac{n\pi \ln r}{\ln r_0} \frac{dr}{r} = 0$ for all n then

$\int_{\ln r_0}^{0} f(e^u) \sin \frac{n\pi u}{\ln r_0} du = 0 \Rightarrow f = 0$

for all n

$$g(r) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi \ln r}{\ln r_0} \quad \& \quad h(r) = \sum_{n=1}^{\infty} d_n \sin \frac{n\pi \ln r}{\ln r_0}$$

$$u(r,0) = \sum_{n=1}^{\infty} \sin \frac{n\pi \ln r}{\ln r_0} (a_n + b_n) = g(r)$$

$$u(r,\beta) = \sum_{n=1}^{\infty} \sin \frac{n\pi \ln r}{\ln r_0} \left(a_n e^{n\pi\beta/\ln r_0} + b_n e^{-n\pi\beta/\ln r_0} \right) = h(r)$$

then

$$\begin{cases} a_n + b_n = c_n \\ a_n e^{n\pi\beta/\ln r_0} + b_n e^{-n\pi\beta/\ln r_0} = d_n \end{cases}$$

solve for a_n & b_n.

Bessel functions

Wave equation in polar coordinates

$$u_{tt} = c^2 (u_{rr} + r^{-1} u_r + r^{-2} u_{\theta\theta})$$

look for solutions of the form $u = R(r)\Theta(\theta)T(t)$

$$T''(t)R(r)\Theta(\theta) = c^2 T(t)\left[R''(r)\Theta(\theta) + r^{-1}\Theta(\theta)R'(r) + r^{-2}R(r)\Theta''(\theta)\right]$$

$$\frac{T''(t)}{c^2 T(t)} = \left[\frac{R''(r)}{R(r)} + \frac{r^{-1}R'(r)}{R(r)} + r^{-2}\frac{\Theta''(\theta)}{\Theta(\theta)}\right] = -\mu^2$$

μ^2: constant

$$T''(t) + c^2\mu^2 T(t) = 0$$

$$r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} = -\mu^2 r^2$$

$$r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} + \mu^2 r^2 = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \nu^2$$

ν^2: constant

$$\begin{cases} \Theta''(\theta) + \nu^2 \Theta(\theta) = 0 \\ r^2 R''(r) + r R'(r) + (\mu^2 r^2 - \nu^2) R(r) = 0 \qquad (\star) \\ T''(t) + c^2 \mu^2 T(t) = 0 \end{cases}$$

Consider $x = \mu r$ and $f(x) = R\left(\frac{x}{\mu}\right)$; what equation does f satisfy?

$$r = \frac{x}{\mu} \qquad dr = \frac{dx}{\mu} \qquad f'(x) = \frac{df}{dx} = \frac{dR}{dr}\frac{1}{\mu}$$

$$f''(x) = \frac{d^2R}{dr^2}\frac{1}{\mu^2} = R''(r)\frac{1}{\mu^2}$$

$$R''(r) = \mu^2 f''(x)$$

$$(\star) = r^2\mu^2 f''(x) + r\mu f'(x) + ((\mu r)^2 - \nu^2) f(x)$$

$$= \boxed{x^2 f''(x) + x f'(x) + (x^2 - \nu^2) f(x) = 0}$$

$$\begin{cases} \Theta''(\theta) + \nu^2 \Theta(\theta) = 0 \\ r^2 R''(r) + r R'(r) + (\mu^2 r^2 - \nu^2) R(r) = 0 \qquad (*) \\ T''(t) + c^2 \mu^2 T(t) = 0 \end{cases}$$

Consider $x = \mu r$ and $f(x) = R\left(\frac{x}{\mu}\right)$; what equation does f satisfy?

$$f'(x) = \frac{1}{\mu} R'\left(\frac{x}{\mu}\right) \qquad f''(x) = \frac{1}{\mu^2} R''\left(\frac{x}{\mu}\right) \qquad r = \frac{x}{\mu}$$

$$\mu f'(x) = R'\left(\frac{x}{\mu}\right) \qquad \mu^2 f''(x) = R''\left(\frac{x}{\mu}\right) \qquad r = \frac{x}{\mu}$$

$(*)$ becomes $\left(\frac{x}{\mu}\right)^2 \mu^2 f''(x) + \left(\frac{x}{\mu}\right) \mu f'(x) + (x^2 - \nu^2) f(x) = 0$

$$\boxed{x^2 f''(x) + x f'(x) + (x^2 - \nu^2) f(x) = 0}$$

Bessel equation of order ν

Solutions are Bessel functions

$$x^2 f''(x) + x f'(x) + (x^2 - \nu^2) f(x) = 0$$

We assume $\nu \in \mathbb{R}$ $\nu \geq 0$, notice that when $\nu \neq 0$ when $x = 0$ then $f(x) = 0$. Look for solutions

$$f(x) = \sum_{j=0}^{\infty} a_j x^{j+b} \qquad a_0 \neq 0 \qquad a_j \text{ \& } b \text{ to be determined}$$

$$\sum_{j=0}^{\infty} a_j \left[(j+b)(j+b-1) x^{j+b} + (j+b) x^{j+b} + x^{j+b+2} - \nu^2 x^{j+b} \right] = 0$$

coefficients for $a_j x^{j+b}$

$$(j+b)(j+b-1) + (j+b) - \nu^2 = (j+b)^2 - \nu^2$$

$$\sum_{j=0}^{\infty} a_j x^{j+b+2} = \sum_{k=2}^{\infty} a_{k-2} x^{k+b} = \sum_{j=2}^{\infty} a_{j-2} x^{j+b}$$

$\uparrow$ $k = j+2$

$$\sum_{j=0}^{\infty} a_j \left[(j+b)^2 - \nu^2 \right] x^{j+b} + \sum_{j=2}^{\infty} a_{j-2} x^{j+b} = 0$$

if $j=0$ $\quad (b^2-\nu^2)a_0 = 0$ $\quad$ if $a_0 \neq 0$ $\quad b = \pm\nu$

$j=1$ $\quad ((1+b)^2-\nu^2)a_1 = 0 \Rightarrow (2b+1)a_1 = 0 \Rightarrow b = -1/2$ or $a_1 = 0$

$j \geq 2$ $\quad ((j+b)^2-\nu^2)a_j + a_{j-2} = 0$

choose $\boxed{b = \nu \geq 0}$ $\Rightarrow a_1 = 0$

$j \geq 2$

$$a_j = -\frac{a_{j-2}}{(j+\nu)^2-\nu^2} = -\frac{a_{j-2}}{j(2\nu+j)}$$

since $a_1 = 0$ $\quad a_{2k+1} = 0$ $\quad$ for all $k \geq 0$

$$a_{2k} = -\frac{a_{2k-2}}{2k(2\nu+2k)} = -\frac{a_{2k-2}}{2^2 k(\nu+k)} = \frac{(-1)^2 a_{2k-4}}{2k(2k-2)(2\nu+2k)(2\nu+2k-2)} = \frac{a_{2k-4}}{2^4 k(k-1)(\nu+k)(\nu+k-1)}$$

$$a_{2k} = \frac{(-1)^k a_0}{2^{2k}\, k!\,(1+\nu)\cdots(k+\nu)}$$

$$f(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+\nu}}{2^{2k}\, k!\,(1+\nu)\cdots(k+\nu)}$$

Standard choice of

$$a_0 = \frac{1}{2^\nu \, \Gamma(\nu+1)} \qquad \text{where} \qquad \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt$$

(Appendix 3)

$$\Gamma(z+1) = z\,\Gamma(z)$$

$$\Gamma(k+\nu+1) = (k+\nu)\,\Gamma(k+\nu) = (k+\nu)(k-1+\nu)\,\Gamma(k-1+\nu) \ldots$$

$$= (k+\nu) \ldots (1+\nu)\,\Gamma(\nu+1)$$

$$a_0 \; \frac{1}{2^{2k}\, k!\, (\nu+1) \ldots (\nu+k)} = \frac{1}{2^{2k+\nu}\, k!\, (\nu+1) \ldots (\nu+k)\,\Gamma(\nu+1)} = \frac{1}{2^{2k+\nu}\, k!\, \Gamma(\nu+k+1)}$$

$$\boxed{J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \cdot \left(\frac{x}{2}\right)^{2k+\nu}}$$

ratio test absolutely convergent

↑ Bessel function of order ν (of 1st kind)

note that $\Gamma(j+1) = j!$ so if $\nu = n$ then

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!} \left(\frac{x}{2}\right)^{2k+n} \qquad n = 0, 1 \ldots$$

define

$$J_{-n}(x) = \sum_{k=n}^{\infty} \frac{(-1)^k}{k!\,(k-n)!} \left(\frac{x}{2}\right)^{2k-n} = \sum_{j=0}^{\infty} \frac{(-1)^{j+n}}{(j+n)!\,j!} \left(\frac{x}{2}\right)^{2j+n}$$

$j = n - k$

$$J_{-n}(x) = (-1)^n J_n(x)$$

if we had taken $b = -\nu$ we would have obtained $J_{-\nu}$

$$J_{-\nu}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k-\nu+1)} \left(\frac{x}{2}\right)^{2k-\nu}$$

if $\nu \notin \mathbb{N}$ near 0 $\quad J_\nu(x) \sim \dfrac{x^\nu}{2^\nu\,\Gamma(\nu+1)}$

if $\nu \in \mathbb{N}$

$$J_{-\nu}(x) \sim \frac{x^{-\nu}}{2^{-\nu}\,\Gamma(-\nu+1)}$$

$J_{-\nu} = (-1)^\nu J_\nu$ not. lin. ind.

} linearly independent solutions of Bessel equation of order ν

How to find a 2nd linearly independent solution of

$(\star)$ $\quad x^2 f''(x) + x f'(x) + (x^2 - \nu^2) f(x) = 0$ when $\nu \in \mathbb{N}$?

If $\nu \notin \mathbb{N}$ Weber function or Bessel function of 2nd kind

$$Y_\nu(x) = \frac{\cos(\nu\pi) J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)} \quad \text{satisfies } (\star)$$

↑

if $\nu = n \in \mathbb{N}$ $\quad Y_n$ is undefined

$$Y_n(x) = \lim_{\nu \to n} Y_\nu(x) \quad (x \neq 0)$$

solution of $(\star)$ with $\nu = n$

$n = 1, 2 \ldots$

$$Y_n(x) \sim -\frac{(n-1)!}{\pi} \left(\frac{x}{2}\right)^{-n} \quad \text{as } x \to 0$$

Y_n & J_n linearly independent

$$Y_0(x) \sim \frac{2}{\pi} \log\left(\frac{x}{2}\right) \quad x \to 0$$

$$J_n(x) \sim \frac{1}{n!} \left(\frac{x}{2}\right)^n \quad x \to 0$$

Summary: the solutions to $x^2 f''(x) + x f'(x) + (x^2 - \nu^2) f(x) = 0$

if $\nu \notin \mathbb{Z}$ $J_\nu(x)$ & $J_{-\nu}(x)$ linearly independent solutions

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu)} \left(\frac{x}{2}\right)^{2k+\nu}$$

if $\nu \in \mathbb{N}$, $\nu = n$ $J_n(x)$ and $Y_n(x)$ linearly independent.

where $$Y_\nu(x) = \frac{\cos(\nu\pi) J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}$$ and

$$Y_n(x) = \lim_{\nu \to n} Y_\nu(x)$$

Solve $x^2 f''(x) + x f'(x) + (x^2 - \nu^2) f(x) = 0 \quad (*)$

Is $\sum a_j x^{j+b}$ a solution? yes $\left| J_\nu(x) = \sum_{k=0}^{\infty} a_{2k} x^{2k+\nu} \right|$ $\nu \geq 0$

in $(*)$ ν and $-\nu$ play the same role

$\left| J_{-\nu}(x) = \sum_{k=0}^{\infty} b_{2k} x^{2k-\nu} \right|$ is a solution.

J_ν & $J_{-\nu}$ are solutions

Q: are J_ν & $J_{-\nu}$ linearly independent?

(1) yes if $\nu \notin \mathbb{N}$

(2) no if $\nu \in \mathbb{N}$ $\quad J_{-n}(x) = (-1)^n J_n(x)$

↓ there has to be another lin. ind sol. from $J_n(x)$

We introduce $y_\nu(x) = \alpha_\nu J_\nu(x) + \beta_\nu J_{-\nu}(x)$ choose α_ν, β_ν

↑ $\nu \notin \mathbb{Z}$

s.t $\lim_{\nu \to n} y_\nu(x) = y_n(x)$ sol lin. ind $J_n(x)$