The Dirichlet Problem

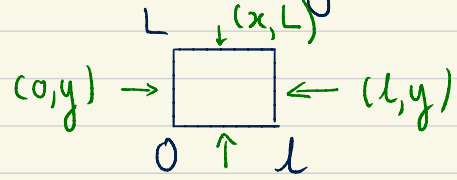
Find the solution to $\begin{cases} \Delta u = \nabla^2 u = 0 & \text{in } D \text{ (region in space)} \\ u = f & \text{on } \partial D \end{cases}$

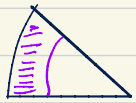
Physical interpretations:

(1) Steady state temperature in D when temperature on ∂D is known.

(2) Electrostatic potential in the charge-free region D when the potential on ∂D is known.

Setting: $D \subset \mathbb{R}^2$: i) D rectangle ii) angular sector





$$D = [0,l] \times [0,L] = \{(x,y) : 0 \le x \le l \text{ & } 0 \le y \le L\}$$

Dirichlet problem becomes

$$u_{xx} + u_{yy} = 0$$

$$u(x,0) = f_1(x) \qquad u(x,L) = f_2(x) \qquad u(0,y) = g_1(y) \qquad u(l,y) = g_2(y)$$

By the Superposition principle ; it suffices to solve the special cases $g_1 = g_2 = 0$ or $f_1 = f_2 = 0$; then the general solution is obtained by adding the 2 solutions. these 2 cases are equivalent as the roles of x & y are symmetric.

Consider

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0,y) = u(l,y) = 0 \end{cases} \qquad u(x,0) = f_1(x) \quad u(x,L) = f_2(x)$$

Separation of variables : look for solutions $u(x,y) = X(x)Y(y)$

then

$$u_{xx} + u_{yy} = X''(x)Y(y) + X(x)Y''(y) = 0$$

thus

$$-\frac{X''}{X} = \frac{Y''}{Y} = \nu^2 \qquad \text{(constant)}$$

$$X'' + \nu^2 X = 0 \qquad X(0) = X(l) = 0 \quad \Longrightarrow \quad \boxed{\nu_n^2 = \left(\frac{n\pi}{l}\right)^2}$$

$$Y'' - \nu^2 Y = 0$$

$$\boxed{X_n = \sin \frac{n\pi x}{l}}$$

$$Y'' - \left(\frac{n\pi}{L}\right)^2 Y = 0 \qquad Y(y) = \alpha_n \cosh\frac{n\pi y}{L} + \beta_n \sinh\frac{n\pi y}{L}$$

we are looking for $u(x,y)$ of the form

$$u(x,y) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(\alpha_n \cosh\frac{n\pi y}{L} + \beta_n \sinh\frac{n\pi y}{L}\right)$$

Expand f_1 & f_2 in their Fourier sine series

$$f_1(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \qquad \& \quad f_2(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

$$u(x,0) = f_1(x) = \sum_{n=1}^{\infty} \alpha_n \sin\frac{n\pi x}{L} \quad \Rightarrow \quad \boxed{\alpha_n = a_n}$$

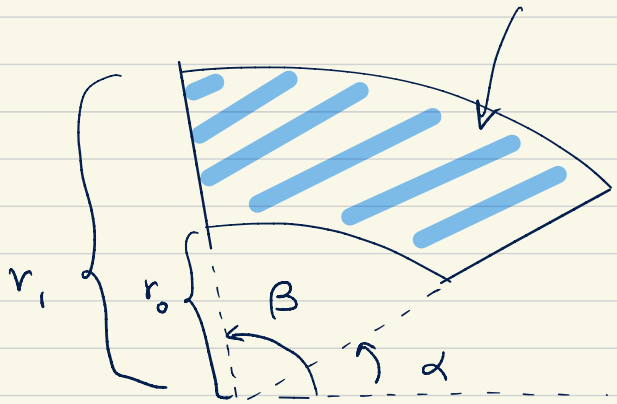
$$u(x,L) = f_2(x) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(\alpha_n \cosh\frac{n\pi L}{L} + \beta_n \sinh\frac{n\pi L}{L}\right)$$

$$b_n = \alpha_n \cosh\frac{n\pi L}{L} + \beta_n \sinh\frac{n\pi L}{L}$$

$$\Rightarrow \boxed{\beta_n = \left(\sinh\frac{n\pi L}{L}\right)^{-1}\left(b_n - a_n \cosh\frac{n\pi L}{L}\right)}$$

Dirichlet problem in polar coordinates

$$S = \{ (r\cos\theta, r\sin\theta) : r_0 \le r \le r_1 \ \text{f} \ \alpha \le \theta \le \beta \}$$



"rectangular region in polar coordinates"

Goal: express the Laplacian of u (C^2) in polar coordinates

$x = r\cos\theta$ $\quad$ $y = r\sin\theta$

$$u_r = \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$(u_{xx} + u_{yy} = 0)$

$$\boxed{u_r = \cos\theta \frac{\partial u}{\partial x} + \sin\theta \frac{\partial u}{\partial y}}$$

(See Appendix 4)

$$u_\theta = \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \boxed{-r\sin\theta \frac{\partial u}{\partial x} + r\cos\theta \frac{\partial u}{\partial y} = u_\theta}$$

$$u_{rr} = u_{xx}\cos^2\theta + 2u_{xy}\cos\theta\sin\theta + u_{yy}\sin^2\theta$$

$$u_{\theta\theta} = r^2(u_{xx}\sin^2\theta - 2u_{xy}\cos\theta\sin\theta + u_{yy}\cos^2\theta) - r(u_x\cos\theta + u_y\sin\theta)$$

$$r^{-2}u_{\theta\theta} = u_{xx}\sin^2\theta - 2u_{xy}\cos\theta\sin\theta + u_{yy}\cos^2\theta - r^{-1}(u_r)$$

$$u_{rr} = u_{xx}\cos^2\theta + 2u_{xy}\cos\theta\sin\theta + u_{yy}\sin^2\theta$$

$$+$$

$$r^{-2}u_{\theta\theta} = u_{xx}\sin^2\theta - 2u_{xy}\cos\theta\sin\theta + u_{yy}\cos^2\theta - r^{-1}u_r$$

$$u_{rr} + r^{-2}u_{\theta\theta} = u_{xx} + u_{yy} - r^{-1}u_r$$

Laplacian polar coordinates

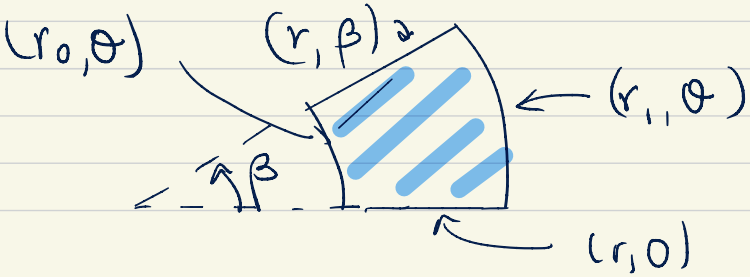
$$\boxed{u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} = u_{xx} + u_{yy} = \Delta u}$$

To solve the Dirichlet problem on S, we separate into 2 different cases : (1) the solution vanishes in the 2 radial pieces

(2) the solution vanishes in the 2 circular pieces

(1) plus by rotation $\alpha = 0$

$$\begin{cases} u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} = 0 & \text{in } S \\ u(r,0) = u(r,\beta) = 0 \qquad u(r_1,\theta) = f(\theta) \qquad u(r_0,\theta) = g(\theta) \end{cases}$$



Separation of variables $u(r,\theta) = R(r)\,\Theta(\theta)$

$$u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} = \left(R'' + r^{-1}R'\right)\Theta(\theta) + r^{-2}R(r)\,\Theta''(\theta) = 0$$

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \nu^2$$

$$\begin{cases} \Theta''(\theta) + \nu^2\Theta(\theta) = 0 \\ \Theta(0) = \Theta(\beta) = 0 \end{cases}$$

$$\boxed{r^2R'' + rR' - \nu^2 R = 0}$$

$$\nu^2 = \left(\frac{n\pi}{\beta}\right)^2$$

Special case of Euler equation

look for solutions of the form $R(r) = r^\lambda \quad (\lambda \in \mathbb{R})$

$$\Theta(\theta) = \sin\frac{n\pi\theta}{\beta}$$

$$r^2R'' + rR' - \nu^2R = \lambda(\lambda-1)r^\lambda + \lambda r^\lambda - \nu^2 r^\lambda = r^\lambda(\lambda^2 - \nu^2) = 0$$

$$\Rightarrow \lambda = \pm\nu$$

$$u(r,\theta) = \sum_{n=1}^{\infty} \sin\frac{n\pi\theta}{\beta}\left(a_n r^{n\pi/\beta} + b_n r^{-n\pi/\beta}\right)$$

solutions to Laplace's equation in S.

To find a_n, b_n go back to the remaining boundary conditions $f(\theta)$, $g(\theta)$ which we expand in their Fourier sine series

$$f(\theta) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi\theta}{\beta} \qquad g(\theta) = \sum_{n=1}^{\infty} d_n \sin \frac{n\pi\theta}{\beta}$$

$$u(r_1, \theta) = \sum_{n=1}^{\infty} \sin \frac{n\pi\theta}{\beta} \left(a_n r_1^{n\pi/\beta} + b_n r_1^{-n\pi/\beta} \right) = f(\theta)$$

$$u(r_0, \theta) = \sum_{n=1}^{\infty} \sin \frac{n\pi\theta}{\beta} \left(a_n r_0^{n\pi/\beta} + b_n r_0^{-n\pi/\beta} \right) = g(\theta)$$

$$\begin{cases} c_n = a_n r_1^{n\pi/\beta} + b_n r_1^{-n\pi/\beta} \\ d_n = a_n r_0^{n\pi/\beta} + b_n r_0^{-n\pi/\beta} \end{cases}$$

solve for a_n & b_n.

Dirichlet problem in the annulus

$$A = \{ (r\cos\theta, r\sin\theta) : r_0 \le r \le r_1 \ ; \ \theta \text{ arbitrary} \}$$



here we look for periodic solutions to

$$\Theta'' + \nu^2 \Theta = 0 \rightsquigarrow \nu^2 = n^2 \ \& \ \Theta(\theta) = e^{\pm in\theta} \qquad n = 0, 1, \dots$$

$$r^2 R'' + r R' - n^2 R = 0 \rightsquigarrow \begin{cases} a_n r^n + b_n r^{-n} & n \neq 0 \\ a_0 + b_0 \log r \end{cases}$$

$r^2 R'' + r R' = 0$ either $R' = 0$

$\therefore$ $\Big\downarrow$ or $R' \neq 0$ $\qquad R(r) = 1$

$$\frac{R''}{R'} = -\frac{1}{r} \rightsquigarrow \log R' = -\log r$$

$$R' = \frac{1}{r} \Rightarrow R = \log r$$

$$u(r,\theta) = (a_0 + b_0 \log r) + \sum_{n=\pm 1,\pm 2} e^{in\theta}(a_n r^n + b_n r^{-n})$$

Dirichlet problem in the disc $D = \{(x,y) : x^2+y^2 \le r_1^2\}$

$= \{(r\cos\theta, r\sin\theta) : r \le r_1\}$

(*) $\begin{cases} u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} = 0 & \text{in } D \\ u(r_1,\theta) = f(\theta) \end{cases}$ ($r_0 \to 0$, u continuous)

$$u(r,\theta) = \sum_{-\infty}^{\infty} c_n e^{in\theta} r^{|n|}$$

and $$u(r_1,\theta) = \sum_{-\infty}^{\infty} c_n r_1^{|n|} e^{in\theta} = f(\theta)$$

$c_n r_1^{|n|}$ Fourier coefficients of f. Let $r_1 = 1$ then

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\, e^{-in\phi}\, d\phi \quad \text{and}$$

$$u(r,\theta) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} r^{|n|} e^{in\theta} \int_{-\pi}^{\pi} f(\phi) e^{-in\phi} d\phi$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi)\, P(r, \theta - \phi)\, d\theta$$

where $P(r, \psi)$ = Poisson kernel

$$P(r,\psi) = \sum_{-\infty}^{\infty} r^{|n|} e^{in\psi} = \sum_{n=0}^{\infty} r^n e^{in\psi} + \sum_{n=1}^{\infty} r^n e^{-in\psi}$$

$$= \sum_{n=0}^{\infty} (re^{i\psi})^n \qquad = \sum_{n=1}^{\infty} (re^{-i\psi})^n$$

$r < 1$

$$= \frac{1}{1 - re^{i\psi}} + re^{-i\psi} \sum_{n=0}^{\infty} r^n e^{-i\psi n}$$

$$= \frac{1}{1 - re^{i\psi}} + \frac{re^{-i\psi}}{1 - re^{-i\psi}} = \frac{1 - re^{-i\psi} + re^{-i\psi}(1 - re^{i\psi})}{(1 - re^{i\psi})(1 - re^{-i\psi})}$$

$$= \frac{1 - r^2}{(1 - re^{i\psi})(1 - re^{-i\psi})} = \boxed{\frac{1 - r^2}{1 - 2r\cos\psi + r^2} = P(r,\psi)}$$

Poisson kernel in unit disc

Poisson integral formula in the unit disc in $\mathbb{R}^2$

$$u(r,\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)} f(\phi)\, d\phi$$

§ 4.4 problem 7 Solve the following Dirichlet problem

$$\Delta u = 0 \quad \text{in} \quad S = \{(r,\theta) : 0 < r_0 \le r \le 1 \,,\ 0 \le \theta \le \beta\}$$

$$u(r_0,\theta) = u(1,\theta) = 0 \qquad u(r,0) = g(r) \qquad u(r,\beta) = h(r)$$

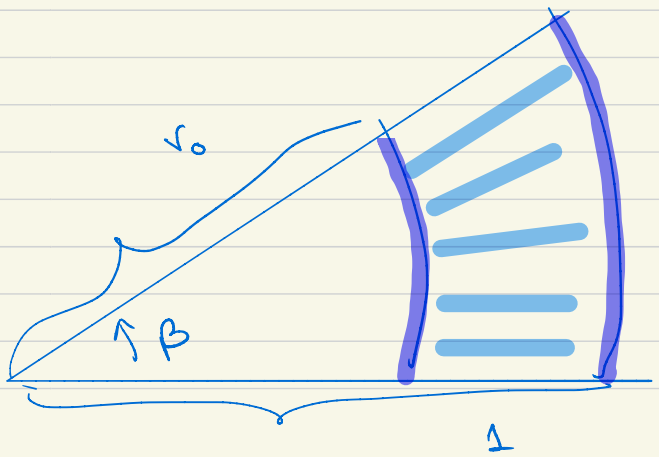
(cf. problem 10 § 3.5)

$u(r,\theta) = R(r)\,\Theta(\theta)$

$$\Theta''(\theta) \pm \nu^2 \Theta(\theta) = 0$$

$$r^2 R''(r) + r R' \mp \nu^2 R = 0$$

$$R(r_0) = 0 \qquad R(1) = 0$$



§ 4.4 problem 7 Solve the following Dirichlet problem

$$\Delta u = 0 \quad \text{in} \quad S = \{(r,\theta) : 0 < r_0 \le r \le 1 \,,\ 0 \le \theta \le \beta\}$$

$$u(r_0,\theta) = u(1,\theta) = 0 \qquad u(r,0) = g(r) \qquad u(r,\beta) = h(r)$$

(cf. problem 10 § 3.5)

$$u(r,\theta) = R(r)\,\Theta(\theta)$$

$$\begin{cases} \Theta''(\theta) + \nu^2 \Theta(\theta) = 0 \\ r^2 R''(r) + r R'(r) + \nu^2 R(r) = 0 \end{cases}$$

$$u(r_0,\theta) = R(r_0)\,\Theta(\theta) = 0$$
$$u(1,\theta) = R(1)\,\Theta(\theta) = 0$$

$$\rightsquigarrow \quad R(r_0) = R(1) = 0$$

Thus we need to solve

$$\begin{cases} r^2 R''(r) + r R'(r) + \nu^2 R(r) = 0 \\ R(r_0) = R(1) = 0 \end{cases}$$

$$\begin{cases} r^2 R''(r) + r R'(r) + \nu^2 R(r) = 0 \\ R(r_0) = R(1) = 0 \end{cases}$$

Change of variable $t = \ln r$ define $T(t) = R(e^t)$

compute the equation T satisfies solve the appropriate boundary data (transform $R(r_0) = R(1) = 0$ onto conditions

$$\begin{cases} T(0) = R(1) = 0 \\ T(\ln r_0) = R(r_0) = 0 \end{cases}$$