Problem 2: Let f be the periodic extension, with period 16, of

$$f(x) = \begin{cases} |x| & \text{for } |x| < 4 \\ 0 & \text{for } 4 < |x| < 8. \end{cases}$$

1. Find the Fourier series for f.
2. What is the sum of the series at $x = 4$. Show your work and/or briefly explain your reasoning.

Problem 3: Consider the following boundary value problem: For $0 \le x \le 4$ and $t > 0$,

$$u_t = u_{xx} - e^{x/2}, \qquad u_x(0,t) = 2, \qquad u_x(4,t) = u(4,t), \qquad \text{and} \qquad u(x,0) = 3.$$

1. Find a steady state solution $u_0(x)$.
2. If the solution of the original problem is $u = u_0 + v$, state the problem that you must solve to find v but **do not solve it**. Justify your answer.

Problem 4: Let $V = P_2(\mathbb{R})$ be the space of real polynomials with degree at most two. Define an inner product on V by

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\,dx.$$

$P_2(x) = 3x^2 - 1$

1. Show that $f_0(x) = 1$ and $f_1(x) = x$ are orthogonal in $[-1, 1]$. Find constants a, b and c so that $f_2(x) = ax^2 + bx + c$ is orthogonal to both f_0 and f_1.
2. Show that if $\langle p, f_i \rangle = 0$ for $i = 0, 1, 2$ then $p = 0$.
3. Show that any polynomial p in V can be written as a liner combination of f_0, f_1 and f_2.

Problem 5: Suppose that f is 2π-periodic, continuous and piecewise smooth and suppose also that f' is piecewise smooth. Assume also that

$$\int_{-\pi}^{\pi} f(x)\,dx = 0$$

1. Show that
$$\int_{-\pi}^{\pi} |f(x)|^2\,dx \le \int_{-\pi}^{\pi} |f'(x)|^2\,dx.$$
2. When does equality hold?

even $f(-x) = f(x)$

odd $f(-x) = -f(x)$

(4) b) $p(x) = ax^2 + bx + c \Rightarrow p = 0$

$\langle p, f_0 \rangle = 0 \qquad \langle p, f_1 \rangle = 0 \qquad \langle p, f_2 \rangle = 0$

(1) $\int_{-1}^{1} (ax^2 + bx + c)\,dx = 0$

$\int_{-1}^{1} x(ax^2 + bx + c)\,dx = 0$ (2) — odd · odd = odd

$\int_{-1}^{1} (3x^2 - 1)(ax^2 + bx + c)\,dx = 0$ (3)

x is odd

$\int_{-1}^{1} (ax^2 + c)\,dx = 0 \Rightarrow \frac{a}{3}x^3 + cx \Big|_{-1}^{1} = \boxed{\frac{2a}{3} + c = 0}$ (1)

$\boxed{c = -\frac{2a}{3} = 0}$

(2) $\int_{-1}^{1} bx^2\,dx = b\int_{-1}^{1} x^2\,dx = 0 \Rightarrow \boxed{b = 0}$ (2)

$\int_{-1}^{1} x^2\,dx \neq 0$

$\Rightarrow \boxed{a = 0}$

(3) $\int_{-1}^{1} (3x^2 - 1)\left(ax^2 - \frac{2a}{3}\right)dx = a\int_{-1}^{1} (3x^2 - 1)\left(x^2 - \frac{1}{3}\right)dx = 0$

$(3x^2 - 1)\left(x^2 - \frac{1}{3}\right)$ even

$\neq 0$

4 c) $\{f_0, f_1, f_2\}$ orthogonal $\left\{\frac{f_0}{\|f_0\|}, \frac{f_1}{\|f_1\|}, \frac{f_2}{\|f_2\|}\right\}$ orthonormal set

$$h = p - \left\langle p, \frac{f_0}{\|f_0\|}\right\rangle \frac{f_0}{\|f_0\|} - \left\langle p, \frac{f_1}{\|f_1\|}\right\rangle \frac{f_1}{\|f_1\|} - \left\langle p, \frac{f_2}{\|f_2\|}\right\rangle \frac{f_2}{\|f_2\|}$$

$\langle h, f_0\rangle = 0$ $\langle h, f_1\rangle = 0$ & $\langle h, f_2\rangle = 0$

4 b) $\Rightarrow$ $h = 0$.

Pb 5 : $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

$$\int_{-\pi}^{\pi} f(x)dx = 0 \Rightarrow a_0 = 0$$

f continuous
f piecewise C^1

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2 dx = \sum_{n=1}^{\infty} a_n^2 + b_n^2$$

$$f'(x) = \sum_{n=1}^{\infty} \cdot\, n a_n \sin nx + n b_n \cos nx$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(x)|^2 dx = \sum_{n=1}^{\infty} n^2 a_n^2 + n^2 b_n^2 = \sum_{n=1}^{\infty} n^2 (a_n^2 + b_n^2)$$

$n \geq 1$ $\quad a_n^2 + b_n^2 \leq n^2 (a_n^2 + b_n^2)$

$\rightsquigarrow f(x) = a_1 \cos x + b_1 \sin x$

equality iff $\boxed{n>1}$ $\quad a_n = b_n = 0$

§ 4.2 problem 3

$$\begin{cases} u_t = k u_{xx} \\ u(x,0) = f(x) \end{cases} \qquad \begin{aligned} u(0,t) &= 0 \\ u_x(l,t) &= A \neq 0 \end{aligned}$$

(heat supplied at a constant rate)

Find a series expansion for $u(x,t)$.

Inhomogeneous boundary conditions

$$u(x,t) = v(x,t) + u_0(x)$$

$$\begin{cases} v_t = k v_{xx} \\ v(0,t) = v_x(l,t) = 0 \\ v(x,0) = f(x) - u_0(x) \end{cases}$$

$$\begin{cases} (u_0)_{xx} = 0 \\ u_0(0) = 0 \\ u_0'(l) = A \end{cases}$$

look for a sol that only depends on x

$$u(x,t) = X(x)\,T(t)$$

$$X''(x) = A X(x)$$

$$\begin{cases} X(0)\,T(t) = 0 \rightsquigarrow X(0) = 0 \\ X'(l)\,T(t) = A \\ T(t) = \text{constant} \end{cases}$$

§ 4.2 problem 3

$$\begin{cases} u_t = k u_{xx} \\ u(x,0) = f(x) \end{cases} \qquad \begin{aligned} u(0,t) &= 0 \\ u_x(l,t) &= A \neq 0 \end{aligned}$$

(heat supplied at a constant rate)

Inhomogeneous boundary conditions

Steady state solution $(u_0)_{xx} = 0$

$u_0(x) = ax + b$

$u_0(0) = b = 0$

$u_x(l) = A = a$

$u_0(x) = Ax$

$u(x,t) = u_0(x) + v(x,t)$

$$\begin{cases} v_t = k v_{xx} \\ v(x,0) = f(x) - Ax \end{cases} \qquad \begin{aligned} v(0,t) &= 0 \\ v_x(l,t) &= 0 \end{aligned}$$

$v(x,t) = X(x)T(t)$

$X(0) = 0$

$X'(l) = 0$

$X'' + \nu^2 X = 0$

$$\begin{cases} X(x) = a\cos\nu x + b\sin\nu x \\ X'(x) = -a\nu\sin\nu x + b\nu\cos\nu x \end{cases}$$

$\rightsquigarrow$ $X(0) = a = 0$ $\qquad X'(l) = b\nu\cos\nu l = 0$ $\qquad l\nu = \left(n + \frac{1}{2}\right)\pi$

$n = 0, 1 \ldots$

$$X(x) = \sin\left(n+\frac{1}{2}\right)\frac{\pi x}{l} \qquad T(t) = e^{-\left(n+\frac{1}{2}\right)^2 \frac{\pi^2}{l^2} kt}$$

if $$b_n = \frac{2}{l}\int_0^l f(x) \sin\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\, dx$$

then $$f(x) = \sum b_n \sin\frac{(n+1/2)\pi x}{l}$$

$\left\{\sin\frac{(n+1/2)\pi x}{l}\right\}_n$ orthogonal family

if $$a_n = \frac{2}{l}\int_0^l Ax \sin\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\, dx ;$$

$$= \frac{2A}{l}\left(-x \frac{1}{(n+\frac{1}{2})\frac{\pi}{l}} \cancel{\cos\left(n+\frac{1}{2}\right)\frac{\pi x}{l}}^{\,0}\right)\Bigg|_0^l + \frac{2A}{(n+\frac{1}{2})\pi}\int_0^l \cos\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\, dx$$

$$= \frac{2Al}{\left((n+\frac{1}{2})\pi\right)^2} \sin\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\Bigg|_0^l = \frac{2Al\,(-1)^n}{\left((n+\frac{1}{2})\pi\right)^2}$$

$$-Ax = \sum_{n=0}^{\infty} \frac{8Al}{(2n+1)^2\pi^2}(-1)^{n+1}$$

$$v(x,t) = \sum_{n=0}^{\infty} \left(b_n - a_n \right) e^{-\left(n+\frac{1}{2}\right)^2 \pi^2 k t / l^2} \sin\left(n+\frac{1}{2}\right)\frac{\pi x}{l}$$

$$v(x,t) = \sum_{n=0}^{\infty} \left(b_n + \frac{(-1)^{n+1} 8Al}{(2n+1)^2 \pi^2} \right) e^{-\left(\frac{n+1/2}{l^2}\right)^2 \pi^2 k t} \sin\left(n+\frac{1}{2}\right)\frac{\pi x}{l}$$

$$u(x,t) = u_0(x) + v(x,t)$$

Inhomogeneous heat equations

$$u_t = k u_{xx} + F(x,t) \qquad u(x,0) = 0 \qquad u(0,t) = u(l,t) = 0$$

Case 1 $F(x,t) = F(x)$, look for a steady-state solution

Solve $k u_0'' + F(x) = 0 \qquad u_0(0) = u_0(l) = 0$

then $u(x,t) = v(x,t) + u_0(x)$ is a solution provided

$$\begin{cases} v_t = k v_{xx} \\ v(0,t) = v(l,t) = 0 \\ v(x,0) = -u_0(x) \end{cases}$$

Case 2: Consider the eigenfunctions ϕ_n of the Sturm-Liouville problem arising from separation of variables in the homogeneous problem $u_t = k u_{xx}$, $u(x,0) = 0$ & $u(0,t) = u(l,t) = 0$.

Expand $F(x,t) = \sum_n \beta_n(t)\, \phi_n(x)$ and look for solutions of the form $u(x,t) = \sum_n b_n(t)\, \phi_n(x)$.

Here $\phi_n(x) = \sin \frac{n\pi x}{l}$.

$u_t = k u_{xx} + F(x,t)$ translates into

$$\sum b_n'(t) \sin\left(\frac{n\pi x}{\ell}\right) = \sum \left[-\left(\frac{n\pi}{\ell}\right)^2 k\, b_n(t) + \beta_n(t) \right] \sin \frac{n\pi x}{\ell}$$

thus

$$b_n'(t) = -k\left(\frac{n\pi}{\ell}\right)^2 b_n(t) + \beta_n(t)$$

$e^{k\left(\frac{n\pi}{\ell}\right)^2 t}$ integrating factor

$$\frac{d}{dt}\left[b_n(t)\, e^{k\left(\frac{n\pi}{\ell}\right)^2 t} \right] = \beta_n(t)\, e^{k\left(\frac{n\pi}{\ell}\right)^2 t}$$

$$b_n(t) = e^{-k\left(\frac{n\pi}{\ell}\right)^2 t} \int_0^t \beta_n(s)\, e^{k\left(\frac{n\pi}{\ell}\right)^2 s}\, ds$$

then

$$u(x,t) = \sum \left(e^{-k\left(\frac{n\pi}{\ell}\right)^2 t} \int_0^t \beta_n(s)\, e^{k\left(\frac{n\pi}{\ell}\right)^2 s}\, ds \right) \sin \frac{n\pi x}{\ell}$$

is a "solution"

§ 4.2 problem 4 $\begin{cases} u_t = k u_{xx} + R \\ u(x,0) = f(x) \end{cases}$ $u(0,t) = 0$, $u_x(l,t) = 0$

Find a series expansion for $u(x,t)$.

the rod generates heat within itself at a constant rate R.

§ 4.2 problem 4 $\begin{cases} u_t = k u_{xx} + R & u(0,t) = 0 \\ u(x,0) = f(x) & u_x(l,t) = 0 \end{cases}$

Find a series expansion for $u(x,t)$ — R independent of t

Steady state solution

$$k(u_0)_{xx} + R = 0 \qquad u_0(0) = 0 \qquad u_0'(l) = 0$$

$$u_0(x) = -\frac{R}{2k}x^2 + ax + b \qquad u_0(0) = 0 \Rightarrow b = 0$$

$$u_0'(x) = -\frac{Rx}{k} + a \qquad u_0'(l) = -\frac{Rl}{k} + a = 0 \qquad a = \frac{Rl}{k}$$

$$\boxed{u_0(x) = -\frac{R}{2k}x^2 + \frac{Rl}{k}x}$$

$$u(x,t) = v(x,t) + u_0(x) \qquad v(0,x) = f(x) - u_0(x)$$

$$v(0,t) = 0 \quad \& \quad v_x(l,t) = 0 \qquad v_t = k v_{xx}$$

$$v(x,t) = X(x)T(t)$$

$$X''(x) + \nu X(x) = 0 \qquad X(0) = 0 \qquad X'(l) = 0$$

$n = 0, 1 \ldots$

$$X(x) = \sin\left(n+\frac{1}{2}\right)\frac{\pi x}{l} \qquad T(t) = e^{-\left(n+\frac{1}{2}\right)^2 \frac{\pi^2}{l^2} kt}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\,dx \quad \& \quad f(x) = \sum b_n \sin\frac{(n+1/2)\pi x}{l}$$

$$a_n = \frac{2}{l}\int_0^l \left(\frac{R}{2k}x^2 - \frac{Rlx}{k}\right)\sin\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\,dx$$

$$= \frac{2R}{lk}\left(\frac{x^2}{2} - lx\right)\frac{-1}{\left(n+\frac{1}{2}\right)\frac{\pi}{l}}\cos\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\Bigg|_0^l \quad (0)$$

$$+ \frac{2R}{kl}\,\frac{1}{\left(n+\frac{1}{2}\right)\frac{\pi}{l}}\int_0^l (x-l)\cos\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\,dx$$

$$= \frac{2R}{\left(n+\frac{1}{2}\right)\pi k}(x-l)\,\frac{1}{\left(n+\frac{1}{2}\right)\frac{\pi}{l}}\sin\left(n+\frac{1}{2}\right)\frac{\pi}{l}x\Bigg|_0^l \quad \curvearrowleft \; 0$$

$$-\frac{2Rl}{k\left(n+\frac{1}{2}\right)^2\pi^2}\int_0^l \sin\left(n+\frac{1}{2}\right)\frac{\pi}{l}x\,dx \quad \curvearrowleft$$

$$+\frac{2Rl^2}{k\left(n+\frac{1}{2}\right)^3\pi^3}\cos\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\Bigg|_0^l = \frac{-2Rl^2}{k\left(n+\frac{1}{2}\right)^3\pi^3}$$

$$v(x,t) = \sum_{n=0}^{\infty} (a_n + b_n)\, e^{-\left(n+\frac{1}{2}\right)^2 \frac{\pi^2}{l^2} kt} \sin\left(n+\frac{1}{2}\right)\frac{\pi x}{l}$$

$$= \sum_{n=0}^{\infty} \left(b_n - \frac{16Rl^2}{k\pi^3(2n+1)^3}\right) e^{-\left(n+\frac{1}{2}\right)^2 \frac{\pi^2}{l^2} kt} \sin\left(n+\frac{1}{2}\right)\frac{\pi x}{l}$$

$$u(x,t) = \frac{R(2lx - x^2)}{2k} + \sum_{n=0}^{\infty} \left(b_n - \frac{16Rl^2}{k\pi^3(2n+1)^3}\right) e^{-\left(n+\frac{1}{2}\right)^2 \frac{\pi^2}{l^2} kt} \sin\left(n+\frac{1}{2}\right)\frac{\pi x}{l}$$

One dimensional wave motion

Before

$$u_{tt} = c^2 u_{xx} \qquad u(0,t) = u(l,t) = 0 \qquad u(x,0) = f(x) \qquad u_t(x,0) = g(x)$$

Inhomogeneous wave equation

$$u_{tt} = c^2 u_{xx} + F(x,t)$$ ← external force affecting the vibration

$$u(0,t) = u(l,t) = 0 \qquad u(x,0) = f(x) \qquad u_t(x,0) = g(x)$$

Vibrations with free ends

$$u_{tt} = c^2 u_{xx}$$

$$u_x(0,t) = u_x(l,t) = 0 \qquad u(x,0) = f(x) \qquad u_t(x,0) = g(x)$$

Mixed boundary conditions

$$u_{tt} = c^2 u_{xx}$$

$$u(0,t) = u_x(l,t) = 0 \qquad u(x,0) = f(x) \qquad u_t(x,0) = g(x)$$

Inhomogeneous wave equation

$$(\star)\begin{cases} u_{tt} = c^2 u_{xx} + F(x,t) \\ u(0,t) = u(l,t) = 0 \qquad u(x,0) = f(x) \qquad u_t(x,0) = g(x) \end{cases}$$

① if F is independent of t find steady state solution to

$c^2 (u_0)_{xx} + F(x) = 0$ with $u_0(0) = u_0(l) = 0$

then $u(x,t) = v(x,t) + u_0(x)$ where

$v_{tt} = c^2 v_{xx}$ $\qquad v(0,t) = v(l,t) = 0$

$v(x,0) = f(x) - u_0(x) \qquad u_t(x,0) = g(x)$ ✓

In the general case we look for solutions of the form

$$v(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\frac{n\pi x}{l}$$ (they satisfy the boundary conditions)

$\{\sin\frac{n\pi x}{l}\}$ complete orthogonal set.

write $F(x,t) = \sum_{n=1}^{\infty} \beta_n(t) \sin\frac{n\pi x}{l}$ and formally compute

$$v_{tt} = c^2 v_{xx} + F(x,t) \Rightarrow \sum_{n=1}^{\infty} b_n''(t) \sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} \left(-c^2\frac{n^2\pi^2}{l^2} b_n(t) + \beta_n(t)\right) \sin\frac{n\pi x}{l}$$

formally

Differential equation for the Fourier coefficients

$$b_n''(t) + \frac{n^2\pi^2c^2}{l^2} b_n(t) = \beta_n(t)$$

(variation of parameters or Laplace transform)

(ode class)

$$b_n(t) = \frac{l}{n\pi c} \int_0^t \sin\left[\frac{\pi n c}{l}(t-s)\right] \beta_n(s)\, ds$$

$$b_n'(t) = \frac{l}{n\pi c} \sin\frac{\pi n c}{l}(t-s)\beta_n(s)\Big|_{s=t} + \frac{l}{n\pi c}\int_0^t \frac{n\pi c}{l}\cos\frac{n\pi c}{l}(t-s)\beta_n(s)\, ds$$

$$v(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\frac{n\pi x}{l}$$

$v(x,0) = 0 \qquad v_t(x,0) = 0$

$$v_{tt} = c^2 v_{xx} + F(x,t) \qquad v(0,t) = v(l,t) = 0 \quad (*)$$

Then $u(x,t) = w(x,t) + v(x,t)$ with

$$w_{tt} = c^2 w_{xx} \qquad w(0,t) = w(l,t) = 0$$

$$w(x,0) = f(x) \quad \& \quad w_t(x,0) = g(x)$$

} done before

is a solution to (*)

(Formal argument)

Vibrations with free ends

$$u_{tt} = c^2 u_{xx}$$

$$u_x(0,t) = u_x(l,t) = 0 \qquad u(x,0) = f(x) \qquad u_t(x,0) = g(x)$$

Find solutions of the form

$$u(x,t) = \frac{1}{2} A_0(t) + \sum_{n=1}^{\infty} A_n(t) \cos \frac{n\pi x}{l}$$ (satisfy boundary condition)

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \qquad g(x) = \frac{1}{2} \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos \frac{n\pi x}{l}$$

Differential equations for the coefficients

$n = 0$ $\quad \frac{1}{2} A_0''(t) = 0 \qquad \frac{1}{2} A_0(0) = \frac{a_0}{2} \qquad \frac{1}{2} A_0'(0) = \frac{1}{2} \alpha_0$

$n \geq 1$ $\quad A_n''(t) = -c^2 \left(\frac{n\pi}{l}\right)^2 A_n(t) \qquad A_n(0) = a_n \qquad A_n'(0) = \alpha_n$

$$A_0(t) = \alpha_0 t + a_0$$

$$A_n''(t) + \left(\frac{n\pi c}{l}\right)^2 A_n(t) = 0 \qquad A_n(0) = a_n \qquad A_n'(0) = \alpha_n$$

$$A_n(t) = a\cos\frac{n\pi c}{l} + b\sin\frac{n\pi c}{l} \qquad A_n(0) = a = a_n$$

$$A_n'(t) = \frac{n\pi c}{l}\left(-a\sin\frac{n\pi c}{l} + b\cos\frac{n\pi c}{l}\right) \qquad A_n'(0) = b\frac{n\pi c}{l} = \alpha_n$$

$$\boxed{A_n(t) = a_n\cos\frac{n\pi c}{l}t + \frac{\alpha_n l}{n\pi c}\sin\frac{n\pi c}{l}t}$$

formal computation

$$u(x,t) = \frac{1}{2}(\alpha_0 t + a_0) + \sum_{n=1}^{\infty}\cos\frac{n\pi x}{l}\left(a_n\cos\frac{n\pi c t}{l} + \frac{\alpha_n l}{n\pi c}\sin\frac{n\pi c t}{l}\right)$$

$\frac{1}{2}(\alpha_0 t + a_0)$: constant

$\sum_{n=1}^{\infty}$: vibratory motion

(if $\alpha_0 = \frac{1}{l}\int_0^l g(x)\,dx \neq 0$

the string continues to move)



Mixed boundary conditions

$$u_{tt} = c^2 u_{xx}$$

$$u(0,t) = u_x(l,t) = 0 \qquad u(x,0) = f(x) \qquad u_t(x,0) = g(x)$$

Separation of variables $\quad u(x,t) = X(x)\,T(t)$

$$X'' + \lambda X = 0 \qquad \lambda = \nu^2 \qquad X(0) = X'(l) = 0$$

$$X(x) = a\cos\nu x + b\sin\nu x \qquad X(0) = a = 0$$

$$X'(x) = b\nu\cos\nu x \;;\; X'(l) = 0 \Rightarrow \cos\nu l = 0 \Rightarrow \nu = \left(n+\tfrac{1}{2}\right)\frac{\pi}{l}$$

$$T'' + c^2\nu^2 T = 0 \qquad T(t) = a_n \cos\left(n+\tfrac{1}{2}\right)\frac{\pi}{l}ct + b_n \sin\left(n+\tfrac{1}{2}\right)\frac{\pi ct}{l}$$

where for n

$$X(x)\,T(0) = \cos\left(n+\tfrac{1}{2}\right)\frac{\pi x}{l}\,a_n$$

$$X(x)\,T'(0) = b_n \cos\left(n+\tfrac{1}{2}\right)\frac{\pi x}{l}\left(n+\tfrac{1}{2}\right)\frac{\pi c}{l}$$

where

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(n+\tfrac{1}{2}\right)\frac{\pi x}{l}$$

$$g(x) = \sum_{n=0}^{\infty} b_n \left(n+\tfrac{1}{2}\right)\frac{\pi c}{l}\cos\left(n+\tfrac{1}{2}\right)\frac{\pi x}{l}$$

§ 4.3 problem 5 : Find the general solution to

$u_{tt} = c^2 u_{xx} - a^2 u \qquad u(0,t) = u(l,t) = 0$ with arbitrary initial conditions, $-au^2$ represents the force of reaction of the medium of the string.

$$u(x,t) = X(x)T(t)$$

$$T''(t)X(x) = c^2 X''(x)T(t) - a^2 X(x)T(t)$$

$$\frac{1}{c^2}\left(\frac{T''}{T} + a^2\right) = \frac{X''(x)}{X(x)} = -\nu^2 \qquad X(x) = a\cos\nu x + b\sin\nu x$$

$$X_n(x) = \sin\frac{n\pi}{l}x$$

$$\frac{T''}{T} = -a^2 - c^2\left(\frac{n\pi}{l}\right)^2 = -\lambda_n^2$$

$$T(t) = a_n \cos\lambda_n t + b_n \sin\lambda_n t$$

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos\lambda_n t + b_n \sin\lambda_n t\right) \sin\frac{n\pi}{l}x.$$