Theorem 1: Consider the regular Sturm-Liouville problem in $[a,b]$

$$(\star)\quad \begin{cases} L(f) + \lambda w f = 0 \\ B_1(f) = B_2(f) = 0 \end{cases} \qquad L(f) = (rf')' + pf$$

. Then

(1) All eigenvalues are real.

(2) Eigenfunctions corresponding to distinct eigenvalues are orthogonal w.r.t the weight w; i.e if $L(f) + \lambda w f = 0$ and $L(g) + \mu w g = 0$ with $\lambda \neq \mu$ then

$$\langle f, g \rangle_w = \int_a^b f(x)\, \overline{g}(x)\, w(x)\, dx = 0$$

(3) The eigenspace for any eigenvalue λ is at most 2-dimensional.

Theorem 2: For a regular Sturm-Liouville problem as in $(\star)$, there is an orthonormal basis $\{\phi_n\}_{n=1}^{\infty}$ of $L^2_w(a,b)$ consisting of eigenfunctions. If λ_n eigenvalue of ϕ_n, then $\lim_{n \to \infty} \lambda_n = +\infty$. If f is C^2 and satisfies $(\star)$ then

$$\sum \langle f, \phi_n \rangle \phi_n \text{ converges uniformly to } f.$$

Pf of Theorem 2 (Problem 5 § 3.5) $f'' + \lambda f = 0 \quad (w=1)$ in $[0, \ell]$

$f'(0) = 0$ & $f'(\ell) = \beta f(\ell)$

Find the normalized eigenfunctions for this problem.

$f'' + \lambda f = 0 \qquad f'(0) = 0 \qquad f'(l) = \beta f(l)$

① $\lambda = 0 \qquad f'' = 0 \qquad f(x) = ax + b \qquad f'(x) = a \qquad f'(0) = 0 = a$

$\beta f(l) = 0$ if $\beta = 0$ then $f(x) = 1$

if $\beta \neq 0 \quad f(l) = 0 = b \Rightarrow$ f not eigenfunction.

$$\boxed{\lambda = 0 \quad \beta = 0 \quad \varphi_0(x) = 1}$$

② $\boxed{\beta = 0} \qquad \lambda \neq 0 \qquad f'(0) = f'(l) = 0$

$f'' + \lambda f = 0$ if $\boxed{\lambda = -\nu^2 \neq 0}$ $\nu \in \mathbb{R}$

then $f(x) = ae^{-\nu x} + be^{\nu x}$; $f'(x) = \nu(-ae^{-\nu x} + be^{\nu x})$

$f'(0) = 0 = \nu(-a + b) \qquad -a + b = 0$

$f'(l) = 0 = \nu(-ae^{-\nu l} + be^{\nu l}) \Rightarrow -a(e^{-\nu l} - e^{\nu l}) = 0$

$\Rightarrow a = 0 = b$ (no eigenfunction)

③ $\boxed{\beta = 0 \text{ if } \lambda = \nu^2}$ $\qquad f(x) = a\cos \nu x + b \sin \nu x$

$f'(x) = -\nu a \sin \nu x + b\nu \cos \nu x$

$f'(0) = b\nu = 0 \Rightarrow b = 0 \qquad f'(l) = -\nu a \sin \nu l = 0 \quad \forall\, a \neq 0$

$\sin \nu l = 0 \quad \Rightarrow \quad \nu l = \pi n \qquad \nu = \frac{\pi n}{l}$

$$\boxed{\beta = 0 \qquad \lambda = \left(\frac{\pi n}{l}\right)^2 \qquad \varphi_n(x) = \cos \frac{\pi n x}{l}}$$

(4) $\lambda = -\nu^2$

i) $\beta > 0$

ii) $\beta < 0$

(5) $\lambda = \nu^2$

i) $\beta > 0$

ii) $\beta < 0$

(4) $\lambda = -\nu^2$ $\quad f'' - \nu^2 f = 0$ $\quad f'(0) = 0$

$$f(x) = ae^{-\nu x} + be^{\nu x} \qquad f'(x) = -a\nu e^{-\nu x} + b\nu e^{\nu x}$$

$$f'(0) = -a\nu + b\nu = 0 \Rightarrow \nu(b-a) = 0 \qquad \boxed{a = b}$$

$$f'(l) = -a\nu e^{-\nu l} + b\nu e^{\nu l} = \beta\left(ae^{-\nu l} + be^{\nu l}\right) \qquad (a \neq 0 \text{ to have eigenfunction})$$

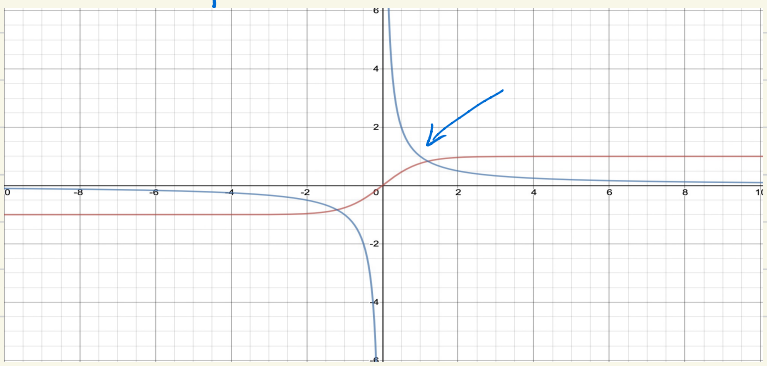
$$= -a\nu\left(e^{-\nu l} - e^{\nu l}\right) = \beta a\left(e^{-\nu l} + e^{\nu l}\right)$$

$$\boxed{\nu\left(e^{\nu l} - e^{-\nu l}\right) = \beta\left(e^{\nu l} + e^{-\nu l}\right)}$$

i) $\beta > 0$ $\quad \nu\left(e^{\nu l} - e^{-\nu l}\right) > 0$ if $\nu > 0$ unique solution of

$$\beta/\nu = \tanh(\nu l)$$

$$\phi(x) = e^{\nu x} + e^{-\nu x}$$



Note: choosing $\nu < 0$ leads to the same solution

(if $\nu < 0$ $e^{\nu l} - e^{-\nu l} < 0$)

ii) $\beta < 0$ $\qquad \underbrace{\nu\left(e^{\nu l} - e^{-\nu l}\right)}_{>0} = \underbrace{\beta}_{<0}\underbrace{\left(e^{\nu l} + e^{-\nu l}\right)}_{>0}$ no solutions

$$f'' - \nu^2 f = 0 \qquad f'(0) = 0 \qquad f'(l) = \beta f(l)$$

$\beta > 0$

$$\phi(x) = e^{\nu x} + e^{-\nu x}$$

where ν is the positive solution of

$$\tanh(\nu l) = \beta / l$$

(5) $f'' + \nu^2 f = 0 \qquad f'(0) = 0$

$$f(x) = a\cos\nu x + b\sin\nu x$$

$$f'(x) = -a\nu\sin\nu x + b\nu\cos\nu x$$

$$f'(0) = b\nu = 0 \Rightarrow b = 0 \qquad f(x) = a\cos\nu x$$

since $a \neq 0$ we may assume $a = 1$.

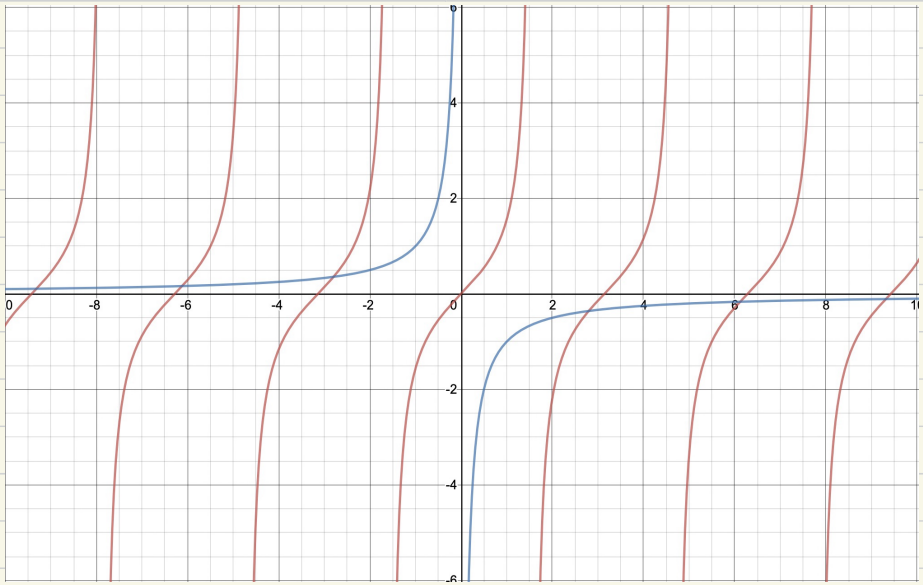
$$f'(l) = \beta f(l)$$

$$-\nu\sin\nu l = \beta\cos\nu l \Rightarrow \tan\nu l = -\beta/\nu \quad (1)$$

i) $\beta > 0$

let $\{\nu_n\}$ all positive solutions to (1)

$\varphi_n(x) = \cos\nu_n x$



if ν solution, $-\nu$ also solution

assume $\nu > 0$

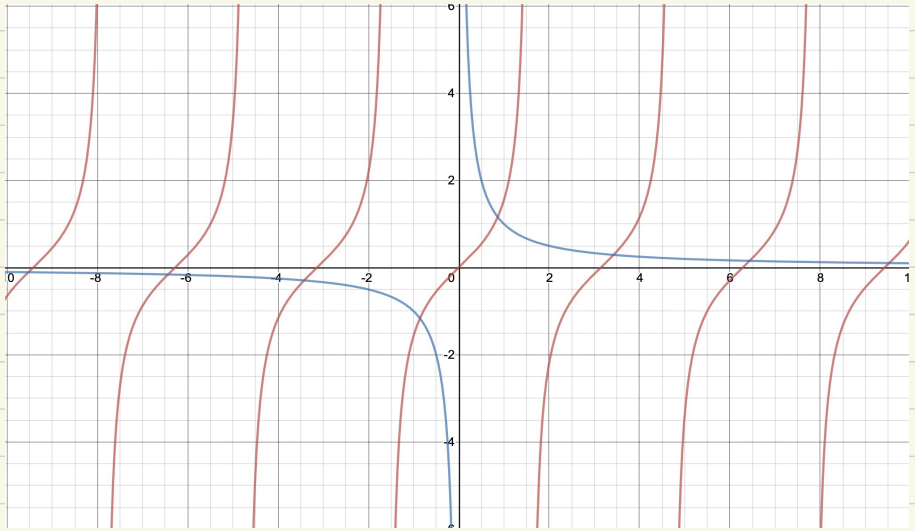
$\cos(\nu x) = \cos(-\nu x)$

ν and $-\nu$ yield same eigenfunction.

ii) $\beta < 0$

let $\{\nu_n\}$ all positive solutions to (1)

$\varphi_n(x) = \cos \nu_n x$



$\tan \nu l = -\beta/\nu$ (1)

if ν solution, $-\nu$ also solution

assume $\nu > 0$

$\cos(\nu x) = \cos(-\nu x)$

ν and $-\nu$ yield same eigenfunction.

Summarizing : eigenfunctions

$\beta = 0 \quad \lambda = 0 \quad \varphi_0(x) = 1$

$\beta = 0 \quad \lambda = \left(\frac{\pi n}{l}\right)^2 \quad \varphi_n(x) = \cos \frac{n\pi x}{l}$

$\beta > 0 \quad \lambda = -\nu_0^2, \quad \nu_0 > 0, \quad \tanh(\nu_0 l) = \beta/\nu_0 \quad \varphi_0^+(x) = e^{\nu_0 x} + e^{-\nu_0 x}$

$\beta > 0 \quad \lambda = \nu_n^2, \quad \nu_n > 0 \quad \tan(\nu_n l) = -\beta/\nu_n \quad \varphi_n^+(x) = \cos \nu_n x$

$\beta < 0 \quad \lambda = \nu_n^2, \quad \nu_n > 0 \quad \tan(\nu_n l) = -\beta/\nu_n \quad \varphi_n^-(x) = \cos \nu_n x$

$\beta = 0 \qquad \lambda = 0 \qquad \varphi_0(x) = 1 \qquad \|\varphi_0\|^2 = \int_0^l \varphi_0(x)^2\, dx = l$

thus $\boxed{\phi_0(x) = \varphi_0(x) / \|\varphi_0\| = 1/\sqrt{l}}$

$\beta = 0 \qquad \lambda = \left(\frac{\pi n}{l}\right)^2 \qquad \varphi_n(x) = \cos\frac{n\pi x}{l}, \quad \|\varphi_n\|^2 = \int_0^l \left(\cos\frac{n\pi x}{l}\right)^2 dx = \frac{l}{2}$

$$\boxed{\phi_n(x) = \frac{\varphi_n(x)}{\|\varphi_n\|} = \sqrt{\frac{2}{l}}\cos n\pi x \qquad \beta = 0 \qquad \lambda = \left(\pi n/l\right)^2}$$

$\beta > 0 \qquad \lambda = -\nu_0^2, \quad \nu_0 > 0, \quad \tanh(\nu_0 l) = \beta/\nu_0 \qquad \varphi_0^+(x) = e^{\nu_0 x} + e^{-\nu_0 x}$

$$\|\varphi_0^+\|^2 = \int_0^l \left(e^{\nu_0 x} + e^{-\nu_0 x}\right)^2 dx = \int_0^l \left(e^{2\nu_0 x} + e^{-2\nu_0 x} + 2\right) dx$$

$$= \frac{1}{2\nu_0}\left(e^{2\nu_0 x} - e^{-2\nu_0 x}\right) + 2x\Big|_0^l = \frac{1}{2\nu_0}\left(e^{+2\nu_0 l} - e^{-2\nu_0 l}\right) + 2l$$

$$= \frac{1}{2\nu_0}\left(e^{\nu_0 l} + e^{-\nu_0 l}\right)\left(e^{+\nu_0 l} - e^{-\nu_0 l}\right) + 2l$$

$$= \frac{4}{2\beta}\tanh\nu_0 l \sinh\nu_0 l \cosh(\nu_0 l) + 2l$$

$$\|\varphi_0^+\|^2 = \frac{4}{2\beta}\tanh\nu_0 l \sinh\nu_0 l \cosh(\nu_0 l) + 2l$$

$$= \frac{2(\sinh\nu_0 l)^2 + 2\beta l}{\beta}$$

$$\phi_0^+ = \frac{\varphi_0^+}{\|\varphi_0^+\|_2}$$

$$\phi_0^+(x) = \frac{2\sqrt{\beta}\cosh\nu_0 x}{\sqrt{2(\sinh\nu_0 l)^2 + 2\beta l}} = \frac{\sqrt{2\beta}}{\sqrt{\beta l + (\sinh\nu_0 l)^2}}\cosh\nu_0 x$$

$\beta > 0$ & $\tanh\nu_0 l = \beta/\nu_0$

$\beta > 0$, $\lambda = \nu_n^2$, $\nu_n > 0$ $\quad \tan(\nu_n l) = -\beta/\nu_n$ $\quad \varphi_n^+(x) = \cos \nu_n x$

$$\|\varphi_n^+\|^2 = \int_0^l (\cos \nu_n x)^2 dx = \int_0^l \frac{1+\cos 2\nu_n x}{2} dx = \frac{x}{2} + \frac{1}{4\nu_n} \sin 2\nu_n x \Big|_0^l$$

$$= \frac{l}{2} + \frac{\sin 2\nu_n l}{4\nu_n} = \frac{l}{2} + \frac{2 \sin \nu_n l \cos \nu_n l}{4\nu_n} = \frac{l}{2} - \frac{\tan \nu_n l \cos \nu_n l \sin \nu_n l}{2\beta}$$

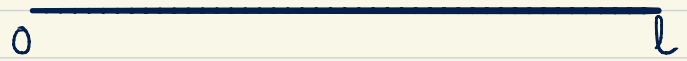
$$= \frac{l}{2} - \frac{(\sin \nu_n l)^2}{2\beta} = \frac{l\beta - (\sin \nu_n l)^2}{2\beta}$$

$$\boxed{\varphi_n^+ = \frac{\varphi_n^+}{\|\varphi_n^+\|} = \sqrt{\frac{2\beta}{l\beta - (\sin \nu_n)^2}} \cos(\nu_n x)} \qquad \beta > 0$$

$\beta < 0$ $\quad \|\varphi_n^-\|^2 = \frac{l}{2} - \frac{(\sin \nu_n l)^2}{2\beta} = \frac{l}{2} + \frac{\sin(\nu_n l)^2}{2|\beta|}$

$$\boxed{\varphi_n^- = \frac{\varphi_n^-}{\|\varphi_n^-\|} = \sqrt{\frac{2|\beta|}{|\beta| l + \sin^2(\nu_n l)}} \cos(\nu_n x)} \qquad \beta < 0$$

Heat flow



Before: heat flow on an insulated rod with conditions at the end

a) ends insulated

b) ends held at temperature 0

- Newton's law of cooling (both ends of the rod are in contact with a medium at temperature 0)

$\alpha > 0$

① $\begin{cases} u_t = k u_{xx} \\ u(x,0) = f(x) \end{cases}$ ② $\begin{cases} u_x(0,t) = \alpha u(0,t) \\ u_x(l,t) = -\alpha u(l,t) \end{cases}$

- Inhomogeneous boundary conditions (ends of the rod at different temperatures 0 & A)

① + $\begin{cases} u(0,t) = 0 \\ u(l,t) = A \end{cases}$ or ① + $\begin{cases} u_x(0,t) = \alpha u(0,t) \\ u_x(l,t) = -\alpha [u(l,t) - A] \end{cases}$

Inhomogeneous heat equations

$$u_t = k u_{xx} + F(x,t) \qquad u(x,0) = 0 \qquad u(0,t) = u(l,t) = 0$$

↑ this models a mechanism that adds or subtracts heat from the rod

– Newton's law of cooling

$$\begin{cases} u_t = k u_{xx} \\ u(x,0) = f(x) \end{cases} \quad \& \quad \begin{cases} u_x(0,t) = \alpha u(0,t) \\ u_x(l,t) = -\alpha u(l,t) \end{cases}$$

($\alpha > 0$ means that if $u(x,t) > 0$ then the temperature is increasing as one crosses the boundary at $x = 0$ from left to right and decreasing as one crosses the boundary at $x = l$ from left to right)

Separation of variables : $u(x,t) = X(x)T(t)$

$$X''(x) = A X(x) \quad \& \quad T'(t) = A k T(t)$$

(check $\nu = 0$)

$A = -\nu^2$ or $A = \nu^2$ $\quad \nu \geq 0$

① $\begin{cases} X'' + \nu^2 X = 0 \\ X'(0) = \alpha X(0) \\ X'(l) = -\alpha X(l) \end{cases}$

② $\begin{cases} X'' - \nu^2 X = 0 \\ X'(0) = \alpha X(0) \\ X'(l) = -\alpha X(l) \end{cases}$

$X(x) = a\cos \nu x + b \sin \nu x$

$X(x) = a e^{\nu x} + b e^{-\nu x}$

① $X(x) = a\cos\sqrt{\nu}x + b\sin\sqrt{\nu}x$

$X'(x) = -a\sqrt{\nu}\sin\sqrt{\nu}x + b\sqrt{\nu}\cos\sqrt{\nu}x$

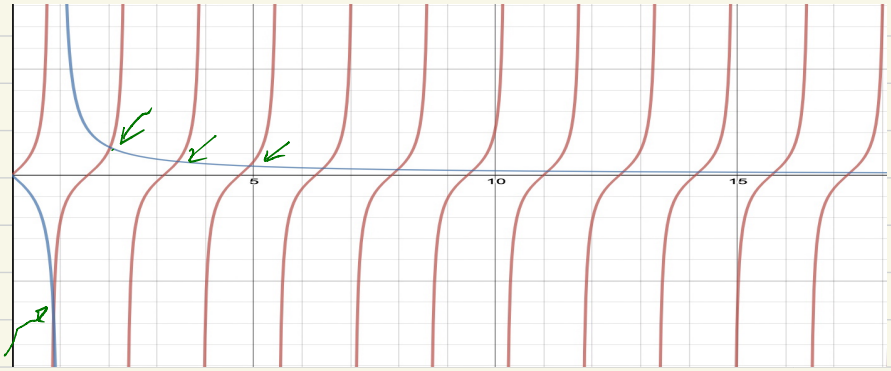
$X'(0) = \alpha X(0) \Rightarrow b\sqrt{\nu} = \alpha a \Rightarrow \boxed{b = \alpha/\sqrt{\nu}\, a} \quad \nu \neq 0$

$X'(l) = -\alpha X(l) \quad -a\sqrt{\nu}\sin\sqrt{\nu}l + b\sqrt{\nu}\cos\sqrt{\nu}l = -\alpha\left(a\cos\sqrt{\nu}l + b\sin\sqrt{\nu}l\right)$

wlog $a = \nu \neq 0$ $\quad -\nu\sin\sqrt{\nu}l + \alpha\sqrt{\nu}\cos\sqrt{\nu}l = -\alpha\sqrt{\nu}\cos\sqrt{\nu}l - \alpha^2\sin\sqrt{\nu}l$

$2\alpha\sqrt{\nu}\cos\sqrt{\nu}l = (\nu^2 - \alpha^2)\sin\sqrt{\nu}l$

$\tan\sqrt{\nu}l = \dfrac{2\alpha\sqrt{\nu}}{\nu^2 - \alpha^2}$



$\{\nu_n\}$ solutions to

ϕ_n function that corresponds to ν_n (normalized)

The normalized eigenfunction corresponding to ν_n^2 is

$$\phi_n(x) = \frac{1}{d_n}(\nu_n \cos\nu_n x + \alpha \sin\nu_n x) \qquad d_n^2 = \frac{1}{2}(\nu_n^2+\alpha^2)\ell + \alpha$$

$$d_n^2 = \int_0^\ell (\nu_n\cos\nu_n x + \alpha\sin\nu_n x)^2 dx = \int_0^\ell \Big(\nu_n^2 \frac{1+\cos 2\nu_n x}{2} + \alpha^2 \frac{1-\cos 2\nu_n x}{2} + 2\nu_n\alpha \sin\nu_n x \cos\nu_n x\Big)dx$$

$$= \frac{1}{2}(\nu_n^2+\alpha^2)\ell + \int_0^\ell \frac{1}{2}(\nu_n^2-\alpha^2)\cos 2\nu_n x + \nu_n\alpha \sin 2\nu_n x$$

$$= \frac{1}{2}(\nu_n^2+\alpha^2)\ell + \underbrace{\frac{1}{2}(\nu_n^2-\alpha^2)\frac{1}{2\nu_n}\sin 2\nu_n x\Big|_0^\ell - \frac{\nu_n\alpha}{2\nu_n}\cos 2\nu_n x\Big|_0^\ell}$$

$$\frac{1}{4\nu_n}(\nu_n^2-\alpha^2)\sin 2\nu_n\ell - \frac{\alpha}{2}(\cos 2\nu_n\ell - 1)$$

$$= \frac{1}{2}(\nu_n^2+\alpha^2)\ell + \frac{\alpha}{2} + \frac{\alpha}{2}\cos 2\nu_n\ell \Big[\underbrace{\frac{\nu_n^2-\alpha^2}{2\nu_n\alpha}}\tan 2\nu_n\ell - 1\Big]$$

$$\frac{2}{1-(\tan\nu_n\ell)^2} - 1 = \frac{1+(\tan\nu_n\ell)^2}{1-(\tan\nu_n\ell)^2} = \frac{1}{(\sin\nu_n\ell)^2-(\cos\nu_n\ell)^2} = \frac{1}{\cos 2\nu_n\ell}$$

(2) $X(x) = ae^{\nu x} + be^{-\nu x}$ $\qquad X'(x) = a\nu e^{\nu x} - b\nu e^{-\nu x}$

$\nu \geq 0$

$X'(0) = a\nu - b\nu = \alpha(a+b)$

$X'(l) = a\nu e^{\nu l} - b\nu e^{-\nu l} = -\alpha(ae^{\nu l} + be^{-\nu l})$

$$\begin{cases} (\alpha-\nu)a + (\alpha+\nu)b = 0 \\ (\alpha+\nu)ae^{\nu l} + b(\alpha-\nu)e^{-\nu l} = 0 \end{cases}$$

needed for solutions other 0.

$$\det \begin{pmatrix} \alpha-\nu & \alpha+\nu \\ (\alpha+\nu)e^{\nu l} & (\alpha-\nu)e^{-\nu l} \end{pmatrix} = (\alpha-\nu)^2 e^{-\nu l} - (\alpha+\nu)^2 e^{\nu l} = 0$$

$$(\alpha^2+\nu^2)(e^{-\nu l} - e^{\nu l}) - 2\alpha\nu(e^{-\nu l} + e^{\nu l}) = 0$$

$$-\frac{2\alpha\nu}{\alpha^2+\nu^2} = \frac{e^{\nu l} - e^{-\nu l}}{e^{\nu l} + e^{-\nu l}} = \tanh(\nu l)$$

no solutions

$-\frac{2\alpha\nu}{\alpha^2+\nu^2} < 0$ $\quad \nu > 0$ & $\alpha > 0$

$\tanh(\nu l) > 0$ $\quad$ bec $\nu > 0$, $l > 0$

① $\begin{cases} u_t = k u_{xx} \\ u(x,0) = f(x) \end{cases}$ & $\begin{cases} u_x(0,t) = \alpha u(0,t) \\ u_x(l,t) = -\alpha u(l,t) \end{cases}$

$\phi_n(x) = \frac{1}{d_n}(\nu_n \cos \nu_n x + \alpha \sin \nu_n x)$ ← orthonormal basis for $L^2(0,l)$

$d_n^2 = \frac{1}{2}(\nu_n^2 + \alpha^2)l + \alpha$ solution of $X'' + \nu_n^2 X = 0$

To solve ① expres $f = \sum \langle f, \phi_n \rangle \phi_n$ and recall

$T' = -k\nu_n^2 T$ $\quad u(x,0) = f(x)$ translates into

$T(0) = \langle f, \phi_n \rangle$

$T(t) = \langle f, \phi_n \rangle e^{-k\nu_n^2 t}$

$$u(x,t) = \sum_{n=1}^{\infty} \langle f, \phi_n \rangle e^{-k\nu_n^2 t} \phi_n(x)$$

$$= \sum_{n=1}^{\infty} \frac{2c_n}{(\nu_n^2 + \alpha^2)l + 2\alpha} e^{-k\nu_n^2 t} (\nu_n \cos \nu_n x + \alpha \sin \nu_n x)$$

$$c_n = \int_0^l f(x)(\nu_n \cos \nu_n x + \alpha \sin \nu_n x)\, dx$$

- Inhomogeneous boundary conditions

(A) $\begin{cases} u_t = k u_{xx} & u(0,t) = 0 \\ u(x,0) = f(x) & u(l,t) = A \end{cases}$ (B) $\begin{cases} u_t = k u_{xx} & u_x(0,t) = \alpha u(0,t) \\ u(x,0) = f(x) & u_x(l,t) = -\alpha [u(l,t) - A] \end{cases}$

I - Find a steady-state solution, $u_0(x)$ solution that satisfies the conditions at $x = 0$ & $x = l$

(A)$_0$ $\begin{cases} (u_0)_{xx} = 0 \\ u_0(0) = 0 \quad u_0(l) = A \end{cases}$ (B)$_0$ $\begin{cases} (u_0)_{xx} = 0 \\ (u_0)_x(0) = \alpha u_0(0), \; (u_0)_x(l) = -\alpha (u_0(l) - A) \end{cases}$

$(u_0)_{xx} = 0 \qquad u_0(x) = ax + b$

(A)$_0$ $\begin{cases} b = 0 \\ al = A \Rightarrow a = \dfrac{A}{l} \end{cases}$ (B)$_0$ $\begin{cases} a = \alpha b \\ a = -\alpha(al + b - A) \\ a = -\alpha a l - a + \alpha A \end{cases}$

(A)$_0$ $\quad u_0(x) = \dfrac{A}{l} x$

$a = \dfrac{\alpha A}{2 + \alpha l} \qquad b = \dfrac{A}{2 + \alpha l}$

(B)$_0$ $\quad u_0(x) = \dfrac{A}{2 + \alpha l} (\alpha x + 1)$

Construct solution of (A) + (B) of the form

$$u(x,t) = u_0(x) + v(x,t)$$

$$\begin{cases} u(0,t) = u_0(0) + v(0,t) \\ u(l,t) = u_0(l) + v(l,t) \end{cases}$$

A A

(A) if $\boxed{\begin{array}{l} v_t = k v_{xx} \\ v(0,t) = v(l,t) = 0 \end{array}}$

$$v(x,0) = u(x,0) - u_0(x)$$

$$= f(x) - \frac{A}{l} x$$

then $u = u_0 + v$ solution of (A)

$$u_t = k u_{xx}$$

$$v_t = k (u_0)_{xx} + k v_{xx} = k v_{xx}$$

$$\begin{cases} u_x(0,t) = (u_0)_x(0) + v_x(0,t) \\ u_x(l,t) = (u_0)_x(l) + v_x(l,t) \end{cases}$$

(B) $v_t = k v_{xx}$

$$v_x(0,t) = \alpha v(0,t)$$

$$v_x(l,t) = u_x(l,t) - (u_0)_x(l)$$

$$= -\alpha (u(l,t) - A) + \alpha (u_0(l) - A)$$

$$= -\alpha (u(l,t) - u_0(l)$$

$$= -\alpha v(l,t)$$

$$v_x(l,t) = -\alpha v(l,t)$$

$$v(x,0) = f(x) - u_0(x)$$

solution to B

§ 4.2 problem 3

$$\begin{cases} u_t = k u_{xx} \\ u(x,0) = f(x) \end{cases} \qquad \begin{aligned} u(0,t) &= 0 \\ u_x(l,t) &= A \neq 0 \end{aligned}$$

(heat supplied at a constant rate)

Find a series expansion for $u(x,t)$.

Inhomogeneous boundary conditions

$$u(x,t) = v(x,t) + u_0(x)$$

$$\begin{cases} v_t = k v_{xx} \\ v(0,t) = v_x(l,t) = 0 \\ v(x,0) = f(x) - u_0(x) \end{cases}$$

$$\begin{cases} (u_0)_{xx} = 0 \\ u_0(0) = 0 \\ u_0'(l) = A \end{cases}$$

look for a sol that only depends on x

$$u(x,t) = X(x)\,T(t)$$

$$X''(x) = A X(x)$$

$$\begin{cases} X(0)\,T(t) = 0 \rightsquigarrow X(0) = 0 \\ X'(l)\,T(t) = A \\ T(t) = \text{constant} \end{cases}$$