Regular Sturm - Liouville problems

Recall we obtained the orthogonal bases $\{\cos nx\}_{n=0}^{\infty}$ and $\{\sin nx\}_{n=1}^{\infty}$ in $L^2(a,b)$ as solutions of the boundary value problems

$$\begin{cases} u''(x) + \lambda^2 u(x) = 0 \\ u'(0) = u'(\pi) = 0 \end{cases} \quad \& \quad \begin{cases} u''(x) + \lambda^2 u(x) = 0 \\ u(0) = u(\pi) = 0 \end{cases}$$

We derived the orthogonal basis $\{e^{inx}\}_{n=-\infty}^{\infty}$ for $L^2(-\pi,\pi)$ by considering periodic functions but they could have been obtained as solutions to the following boundary value problem:

$$\begin{cases} u''(x) + \lambda^2 u(x) = 0 \\ u(-\pi) = u(\pi) \ \& \ u'(-\pi) = u'(\pi) \end{cases}$$

A large class of boundary value problems on $[a,b]$ lead to orthogonal bases for $L^2(a,b)$.

GOAL: Investigate this further.

$S: \mathcal{D}_S \longrightarrow L^2(0,\pi)$

$S(u) = u''$ $\qquad \mathcal{D}_S = \{ u \in L^2(0,\pi) : u'(0) = u'(\pi) = 0 \}$

$T(u) = u''$ $\qquad \mathcal{D}_T = \{ u \in L^2(0,\pi) : u(0) = u(\pi) = 0 \}$

$u'' + \lambda^2 u = 0$ $\qquad S(u) = -\lambda^2 u$

Linear algebra $A \in \mathbb{C}^{k \times k}$

$Au = -\lambda^2 u$ $\qquad$ if $u \neq 0$ $\quad u$ eigenvector

$-\lambda^2$ eigenvalue.

A linear transformation $T : \mathbb{C}^k \longrightarrow \mathbb{C}^k$ is called **self adjoint** or **Hermitian** if

$$\langle Ta, b \rangle = \langle a, Tb \rangle \qquad \forall a, b \in \mathbb{C}^k$$

if $T = (T_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le n}}$ self adjoint means $\overline{T_{ij}} = T_{ji}$

Spectral theorem : If T is self adjoint there is orthonormal basis of $\mathbb{C}^k$ consisting of eigenvectors for T.

Goal: find an analogue of this theorem for differential operators acting on $L^2(a,b)$.

Assume

$$\mathcal{D}_S, \mathcal{D}_T \subset L^2(a,b)$$

$$S : \mathcal{D}_S \longrightarrow L^2(a,b)$$
$$T : \mathcal{D}_T \longrightarrow L^2(a,b)$$

linear operators

We say that S and T are adjoint to each other if

$$\langle S(f), g \rangle = \langle f, T(g) \rangle \quad \text{for all } f \in \mathcal{D}_S \text{ and } g \in \mathcal{D}_T$$

S is called **self-adjoint** or **hermitian** if

$$\langle S(f), g\rangle = \langle f, S(g)\rangle \quad \text{for all } f, g \in \mathcal{D}_S.$$

Suppose: L second-order linear differential operator

$$L(f) = rf'' + qf' + pf$$

r, q, p real functions of class C^2 on $[a,b]$

$r(x) \neq 0$ for all $x \in [a,b]$ ($r(x) > 0$ w.log)

Q: What is the adjoint of L?

We are looking for an operator L^* such that

$$\langle L(f), g\rangle = \langle f, L^*(g)\rangle$$

$$\langle L(f), g\rangle = \int_a^b rf''\bar{g}\,dx + \int_a^b pf'\bar{g}\,dx + \int_a^b pf\bar{g}\,dx$$

$$\int_a^b rf''\bar{g}\,dx \overset{\text{integration by parts}}{=} r\bar{g}f'\Big|_a^b - \int_a^b f'(r\bar{g})'\,dx \overset{\text{integration by parts}}{=} \left[r\bar{g}f' - f(r\bar{g})'\right]_a^b + \int_a^b f(r\bar{g})''\,dx$$

$$\int_a^b q f' \bar{g}\, dx \;=\; q\bar{g} f \Big|_a^b - \int_a^b f (q\bar{g})'\, dx$$

integration by parts

$$\langle L(f), g\rangle = \int_a^b f \underbrace{\left[(r\bar{g})'' - (q\bar{g})' + p\bar{g} \right]}_{\overline{L^*(g)}} dx + \left[r f' \bar{g} - f (r\bar{g})' + q f \bar{g} \right]_a^b$$

(recall r, q, p real)

L^* the formal adjoint of L is defined by

$$L^*(g) = (rg)'' - (qg)' + pg = rg'' + (2r' - q) g' + (r'' - q' + p) g$$

We say that L is formally self-adjoint if $L^* = L$

$$L(g) = rg'' + qg' + pg \quad \text{&} \quad L^*(g) = rg'' + (2r' - q) g' + (r'' - q' + p) g$$

Comparing coefficients for $L = L^*$ we need

$$q = 2r' - q \implies q = r'$$

$$p = r'' - q' + p \implies r'' = q' \implies \boxed{q = r'}$$

If L is formally self-adjoint

$$\langle L(f), g\rangle = \langle f, L(g)\rangle + \underbrace{\left[rf'\bar{g} - f(r\bar{g})' + qf\bar{g}\right]_a^b}$$

find the space $\mathcal{D}_L$ where this is 0

L self-adjoint

$$\langle L(f), g\rangle = \langle f, L(g)\rangle$$

Recall L formally self adjoint $\quad r' = q$

Thus

$$\boxed{L(f) = rf'' + r'f' + pf = (rf')' + pf}$$

and the boundary term

$$[rf'\bar{g} - f(r\bar{g})' + qf\bar{g}]_a^b$$

becomes

$$[rf'\bar{g} - fr'\bar{g} - fr\bar{g}' + r'f\bar{g}]_a^b = [r(f'\bar{g} - f\bar{g}')]_a^b$$

Lagrange identity: If L is formally self-adjoint then

$$\langle L(f), g\rangle = \langle f, L(g)\rangle + \underbrace{[r(f'\bar{g} - f\bar{g}')]_a^b}$$

Question: How do we obtain that L is self-adjoint?

(we want a space $\mathcal{D}_L$ where the operator is self-adjoint)

Examples of boundary conditions :

① $f'(a) = c_1 f(a)$ & $f'(b) = c_2 f(b)$ with $c_1, c_2 \in \mathbb{R}$

if f and g satisfy ① then

$$\left[r(f'\bar{g} - f\bar{g}') \right]_a^b = r(b)\left(f'(b)\bar{g}(b) - f(b)\bar{g}'(b) \right)$$

$$\left(c_2 f(b)\bar{g}(b) - f(b)\overline{c_2 g(b)} \right) \qquad c_2 = \bar{c}_2$$

$$- r(a)\left(f'(a)\bar{g}(a) - f(a)\bar{g}'(a) \right)$$

$$c_1 f(a)\bar{g}(a) - f(a)\overline{c_1 g(a)} = 0 \qquad c_1 \overline{g(a)}$$

② Periodic boundary conditions

$f(a) = f(b)$ and $f'(a) = f'(b)$

under the assumption that $r(a) = r(b)$

$$\left[r(f'\bar{g} - f\bar{g}') \right]_a^b = r(b)\left(f'(b)\bar{g}(b) - f(b)\bar{g}'(b) \right)$$

$$- r(a)\left(f'(a)\bar{g}(a) - f(a)\bar{g}'(a) \right)$$

In general (independent) conditions of the form

$$(*)\quad \begin{cases} B_1(f) = \alpha_1 f(a) + \alpha_1' f'(a) + \beta_1 f(b) + \beta_1' f'(b) = 0 \\ B_2(f) = \alpha_2 f(a) + \alpha_2' f'(a) + \beta_2 f(b) + \beta_2' f'(b) = 0 \end{cases}$$

with $\alpha_1, \alpha_1', \alpha_2, \alpha_2', \beta_1, \beta_1', \beta_2, \beta_2'$ real are called self adjoint if

$$\left[r(f'\bar{g} - f\bar{g}') \right]_a^b = 0 \quad \text{for all } f \text{ and } g \text{ satisfying } (*)$$

A regular Sturm-Liouville problem on the interval $[a,b]$ is specified by the following data:

1. a formally self-adjoint operator $L(f) = (rf')' + pf$ with r, r' and p real and continuous on $[a,b]$ and $r > 0$ on $[a,b]$
2. a set of self-adjoint boundary conditions for L,
   $B_1(f) = B_2(f) = 0$
3. a positive continuous function w on $[a,b]$

Goal : Find all solutions of the boundary value problem

$$(**)\quad \begin{cases} Lf + \lambda w f = 0 \\ B_1(f) = B_2(f) = 0 \end{cases} \quad \text{ie} \quad (rf')' + pf + \lambda w f = 0, \quad \lambda \in \mathbb{C}$$

λ = eigenvalue for the Sturm - Liouville problem
corresponding non - trivial solutions are eigenfunctions

Remark : If f and g satisfy $(**)$ then $c_1 f + c_2 g$ also satisfies $(**)$; the space of solutions of $(**)$ is the eigenspace for λ.

For $w > 0$ as above we define on $L^2[a,b]$ the weighted inner product

$$\langle f, g \rangle_w = \int_a^b f(x)\bar{g}(x)w(x)dx = \langle wf, g \rangle = \langle f, wg \rangle$$

$$L^2_w(a,b) = \left\{ f : (a,b) \to \mathbb{C} \; : \; \int_a^b |f|^2(x)w(x)dx < \infty \right\}$$

§ 3.4 problem 2 : Find constants a, b, A, B, C such that $f_0(x) = 1$, $f_1(x) = ax + b$ and $f_2(x) = Ax^2 + Bx + C$ are an orthonormal set in $L^2_w(0,\infty)$ where $w(x) = e^{-x}$.

Hint : $\int_0^\infty x^n e^{-x}\,dx = n!$. Here

$$L^2_w(0,\infty) = \left\{ f : \int_0^\infty |f(x)|^2 w(x)\,dx < \infty \right\}$$

$$\langle f_0, f_1 \rangle = \int_0^\infty (ax+b)e^{-x}\,dx = a\int_0^\infty xe^{-x} + b\int_0^\infty e^{-x}\,dx$$

$$\langle f_0, f_1 \rangle = \boxed{a + b = 0}$$

$$\langle f_0, f_2 \rangle = \int_0^\infty (Ax^2 + Bx + C)e^{-x}\,dx = \boxed{2A + B + C = 0}$$

$$\langle f_1, f_2 \rangle = \int_0^\infty (ax+b)(Ax^2+Bx+C)e^{-x}\,dx = bC\int_0^\infty e^{-x}\,dx$$

$$+ aA\int_0^\infty x^3 e^{-x}\,dx + (aB+bA)\int_0^\infty x^2 e^{-x}\,dx + (bB+aC)\int_0^\infty xe^{-x}\,dx$$

$$\boxed{6aA + 2(aB+bA) + (bB+aC) + bC = 0}$$

$$a+b=0 \qquad 2A+B+C=0 \quad (2)$$

$$6aA+2(aB+bA)+(bB+aC)+bC=0$$

(1) $\Rightarrow \boxed{a=-b} \longrightarrow 6aA+2(aB-aA)+(-aB+aC)-aC=0$

$$4aA+aB=\boxed{a(4A+B)=0} \quad (3)$$

$$\langle f_1, f_1\rangle = 1 = \int_0^\infty (ax+b)^2 e^{-x}\,dx = a^2\int_0^\infty x^2 e^{-x}dx + 2ab\int_0^\infty x e^{-x}dx + b^2\int_0^\infty e^{-x}dx$$

$$\boxed{2a^2+2ab+b^2=1} \quad (4)$$

$$\langle f_2, f_2\rangle = 1 = \int_0^\infty (Ax^2+Bx+C)^2\,dx = \int_0^\infty (A^2x^4+B^2x^2+C^2+2ABx^3+2ACx^2+2BCx)e^{-x}dx$$

$$= 4!\,A^2 + B^2\,2! + C^2 + 3!(2AB) + 4AC + 2BC = 1$$

$$\boxed{24A^2+2B^2+C^2+12AB+4CA+2BC=1} \quad (5)$$

(6)

(1) & (4) $\boxed{a = -b}$ & $2a^2 + 2ab + b^2 = 2a^2 - 2a^2 + a^2 = 1$ $\boxed{a^2 = 1}$

Choose $a = 1$ then $b = -1$ and $\boxed{f_1(x) = x - 1}$

(3) & (6) $a(4A + B) = 0 \Longrightarrow 4A + B = 0 \Rightarrow \boxed{B = -4A}$ (7)

(2) & (5) $2A + B + C = 0$ & $24A^2 + 2B^2 + C^2 + 12AB + 4CA + 2BC = 1$

$\Downarrow$

$(2A + B + C)^2 = 4A^2 + B^2 + C^2 + 4AB + 4AC + 2BC$

$\boxed{20A^2 + B^2 + 8AB = 1}$ (8)

(7) & (8) $20A^2 + 16A^2 - 32A^2 = 4A^2 = 1$ $\boxed{A^2 = \frac{1}{4}}$

Choose $A = \frac{1}{2}$, $B = -2$ and using (2)

$C = -B - 2A = 2 - 1 = 1$

$\boxed{f_2(x) = \frac{1}{2}x^2 - 2x + 1}$

§ 3.4 problem 7 What is the best approximation in norm to the function $f(x) = x$ on $[0, \pi]$ among all functions of the form

a) $a_0 + a_1 \cos x + a_2 \cos 2x$; b) $b_1 \sin x + b_2 \sin 2x$, c) $a \cos x + b \sin x$

a) orthonormal functions $\frac{1}{\sqrt{\pi}}$; $\sqrt{\frac{2}{\pi}} \cos x$, $\sqrt{\frac{2}{\pi}} \cos 2x$

$$\langle f , \frac{1}{\sqrt{\pi}} \rangle = \frac{1}{\sqrt{\pi}} \int_0^\pi x \, dx = \frac{\pi^2}{2\sqrt{\pi}} = \frac{\pi\sqrt{\pi}}{2}$$

$$\langle f, \sqrt{\frac{2}{\pi}} \cos x \rangle = \sqrt{\frac{2}{\pi}} \int_0^\pi x \cos x \, dx = \sqrt{\frac{2}{\pi}} \underbrace{x \sin x \Big|_0^\pi}_{0} - \sqrt{\frac{2}{\pi}} \int_0^\pi \sin x \, dx$$

$$= + \sqrt{\frac{2}{\pi}} \cos x \Big|_0^\pi = - \sqrt{\frac{2}{\pi}} (1 + 1) = -\frac{2\sqrt{2}}{\sqrt{\pi}}$$

$$\langle f , \sqrt{\frac{2}{\pi}} \cos 2x \rangle = \sqrt{\frac{2}{\pi}} \int_0^\pi x \cos 2x \, dx = \sqrt{\frac{2}{\pi}} \underbrace{x \frac{\sin 2x}{2} \Big|_0^\pi}_{0} - \frac{\sqrt{2}}{2\sqrt{\pi}} \int_0^\pi \sin 2x \, dx$$

$$= - \frac{1}{\sqrt{2\pi}} \left(- \frac{1}{2} \cos 2x \right) \Big|_0^\pi = 0$$

a) $\frac{\pi\sqrt{\pi}}{2} \cdot \frac{1}{\sqrt{\pi}} - 2\sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{2}{\pi}} \cos x = \frac{\pi}{2} - \frac{4}{\pi} \cos x$ (#2 table 1)

b) $2 \sin x - 2 \frac{\sin 2x}{2} = 2\sin x - \sin 2x$ (#1 table 1)

c) $-\frac{4}{\pi} \cos x + 2 \sin x$ (combining a) & b))

Theorem 1: Consider the regular Sturm-Liouville problem in $[a,b]$

$$(\star)\quad \begin{cases} L(f) + \lambda w f = 0 \\ B_1(f) = B_2(f) = 0 \end{cases} \qquad L(f) = (rf')' + pf$$

. Then

(1) All eigenvalues are real.

(2) Eigenfunctions corresponding to distinct eigenvalues are orthogonal w.r.t the weight w; i.e if $L(f) + \lambda w f = 0$ and $L(g) + \mu w g = 0$ with $\lambda \neq \mu$ then

$$\langle f, g \rangle_w = \int_a^b f(x)\, \overline{g}(x)\, w(x)\, dx = 0$$

(3) The eigenspace for any eigenvalue λ is at most 2-dimensional.

Theorem 2: For a regular Sturm-Liouville problem as in $(\star)$, there is an orthonormal basis $\{\phi_n\}_{n=1}^{\infty}$ of $L^2_w(a,b)$ consisting of eigenfunctions. If λ_n eigenvalue of ϕ_n, then $\lim_{n \to \infty} \lambda_n = +\infty$. If f is C^2 and satisfies $(\star)$ then

$$\sum \langle f, \phi_n \rangle \phi_n \text{ converges uniformly to } f.$$

Pf: ① all eigenvalues are real : λ eigenvalue with eigenfunction f

$$L(f) + \lambda w f = 0 \quad ; \text{ recall } \quad \langle Lf, g\rangle = \langle f, Lg\rangle$$

thus
$$\langle Lf, f\rangle = -\langle \lambda w f, f\rangle = -\lambda \|f\|_w^2$$

L self-adjoint $\Big\downarrow$ $=$
$$\langle f, Lf\rangle = -\langle f, \lambda w f\rangle = -\bar{\lambda}\langle f, wf\rangle = -\bar{\lambda}\langle wf, f\rangle = -\bar{\lambda}\|f\|_w^2$$

$$\Rightarrow \quad \lambda \|f\|_w^2 = \bar{\lambda}\|f\|_w^2 \Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda \in \mathbb{R}.$$

② $Lf + \lambda w f = 0$ & $Lg + \mu w g = 0$ with f & g non-zero

$$\lambda \langle f, g\rangle_w = \langle \lambda w f, g\rangle = -\langle Lf, g\rangle = -\langle f, Lg\rangle = +\langle f, \mu w g\rangle$$

($\uparrow$ L self-adjoint)

eigenvalues real $\rightsquigarrow$ $= \mu \langle f, g\rangle_w$

$$\Rightarrow \quad (\lambda - \mu)\langle f, g\rangle_w = 0 \quad \Rightarrow \quad \langle f, g\rangle_w = 0$$

③ check

Pf of Theorem 2 (Problem 5 § 3.5) $f'' + \lambda f = 0 \quad (w=1)$ in $[0,\ell]$

$f'(0) = 0$ & $f'(\ell) = \beta f(\ell)$

Find the normalized eigenfunctions for this problem.

(1) Can you find functions satisfying the boundary conditions with $\lambda = 0$?

$f'' = 0 \qquad f'(0) = 0 \qquad f'(\ell) = \beta f(\ell)$

(2) $\lambda \in \mathbb{R} \qquad \lambda = \nu^2 \quad$ or $\quad \lambda = -\nu^2$