Problem 10c. § 3.3 Evaluate $\sum_{n=1}^{\infty} \frac{n^2}{(n^2+1)^2}$ applying Parseval's equation to one of the Fourier expansion in Table 1 § 2.1

Problem 6 § 3.4) Find an example of a sequence $\{f_n\}$ in $L^2(0,\infty)$ such that $f_n \to 0$ uniformly but $f_n \not\to 0$ in norm. $n \to 0$

20. $f(\theta) = \sinh\theta \qquad -\pi < \theta < \pi$ ; Fourier series :

$$\frac{2\sinh\pi}{\sqrt{\pi}} \sum_{1}^{\infty} \frac{(-1)^{n+1} n \sin n\theta}{(n^2+1)\sqrt{\pi}}$$

thus by Parseval's equality

$$\left\{ \frac{\sin n\theta}{\sqrt{\pi}} \right\} \text{ orthonormal in } L^2([-\pi,\pi])$$

$$\|f\|^2 = \frac{4(\sinh\pi)^2}{\pi} \sum_{1}^{\infty} \frac{n^2}{(n^2+1)^2}$$

$$\int_{-\pi}^{\pi} \left( \frac{e^{\theta} - e^{-\theta}}{2} \right)^2 d\theta = \int_{-\pi}^{\pi} \frac{e^{2\theta} + e^{-2\theta} - 2}{2}\, d\theta = \frac{1}{2}e^{2\theta} - \frac{1}{2}e^{-2\theta} - \theta \Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2}e^{2\pi} - \frac{1}{2}e^{-2\pi} - \pi - \left( \frac{1}{2}e^{-2\pi} - \frac{1}{2}e^{2\pi} + \pi \right)$$

$$= e^{2\pi} - e^{-2\pi} - 2\pi = \frac{1}{\pi}\left(e^{\pi} - e^{-\pi}\right)^2 \sum_{n=1}^{\infty} \frac{n^2}{(n^2+1)^2}$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{n^2}{(1+n^2)^2} = \frac{\left(e^{\pi} - e^{-\pi}\right)^2 + 2 - 2\pi}{\left(e^{\pi} - e^{-\pi}\right)^2}\,\pi}$$

Parseval $\quad f = \sum \langle f, \phi_n \rangle \phi_n \quad \{\phi_n\}$ o.n. basis

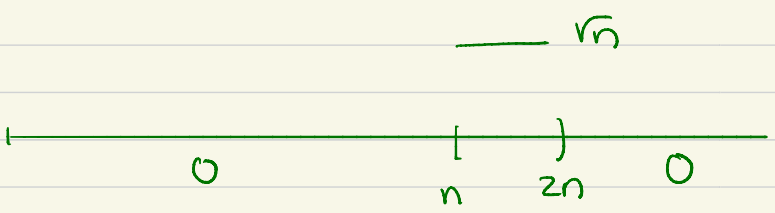
$$\|f\|^2 = \sum |\langle f, \phi_n \rangle|^2 \qquad [-\pi, \pi]$$

$f(\theta) = \sinh \theta \qquad \dfrac{2 \sinh \pi}{\pi} \sum (-1)^{n+1} \dfrac{n \sin n\theta}{n^2+1}$

$$\int_{-\pi}^{\pi} (\sin n\theta)^2 \, d\theta = \pi \qquad \phi_n = \frac{\sin n\theta}{\sqrt{\pi}}$$

Problem 6 § 3.4 Find an example of a sequence $\{f_n\}$ in $L^2(0,\infty)$ such that $f_n \to 0$ uniformly but $f_n \not\to 0$ in norm

$$f_n(x) = \begin{cases} \frac{\sqrt{n}}{x} & n < x < 2n \\ 0 & \text{otherwise} \end{cases}$$




note $|f_n(x)| \le \frac{1}{\sqrt{n}}$ $\quad f_n(x) \to 0$ uniformly

$$\int_0^\infty |f_n(x)|^2\,dx = \int_n^{2n} \frac{n}{x^2}\,dx = n\left(-\frac{1}{x}\right)\Big|_n^{2n} = n\left(-\frac{1}{2n} + \frac{1}{n}\right) = \frac{1}{2}$$

$\not\to 0$

how did I think of it

want $\tilde{f}_n$ ( $|\tilde{f}_n| \le \frac{a_n}{n} \to 0$

$$\tilde{f}_n(x) = \begin{cases} a_n/x & n < x < 2n \\ 0 & \text{otherwise} \end{cases}$$

$$\int_0^\infty |\tilde{f}_n|^2\,dx = a_n^2 \underbrace{\int_n^{2n} \frac{dx}{x^2}}_{\to 0}$$

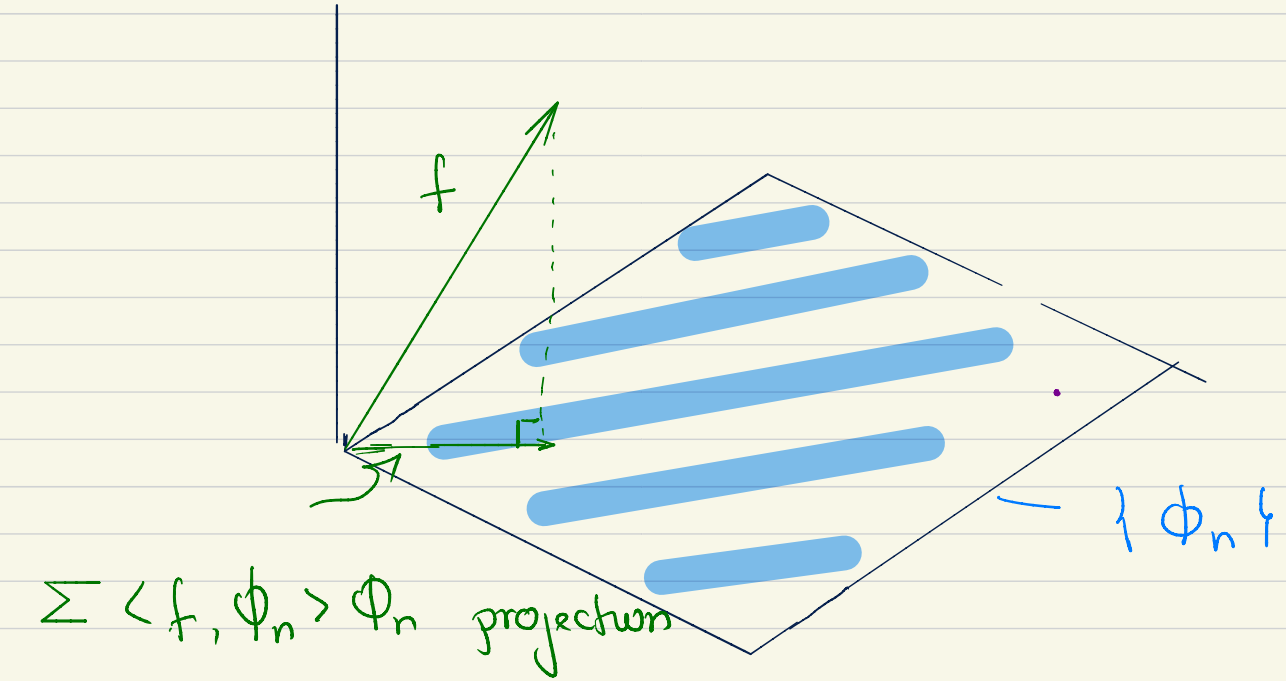
# Best approximations in $L^2(a,b)$

**Theorem:** If $\{\phi_n\}$ is an orthonormal set in $L^2(a,b)$ and $f \in L^2(a,b)$ then

$$\| f - \sum \langle f, \phi_n \rangle \phi_n \| \leq \| f - \sum c_n \phi_n \|$$

for all choices of $\{c_n\}$ with $\sum |c_n|^2 < \infty$. Equality holds only when $c_n = \langle f, \phi_n \rangle$ for all $n$.

Proof



Proof: $$f - \sum c_n \phi_n = \underbrace{f - \sum \langle f, \phi_n \rangle \phi_n}_{g} + \sum (\langle f, \phi_n \rangle - c_n) \phi_n$$

note that $\langle g, \phi_m \rangle = 0$ for all $m$ thus by Pythagorean theorem

$$\| f - \sum c_n \phi_n \|^2 = \| f - \sum \langle f, \phi_n \rangle \phi_n \|^2 + \underbrace{\sum |\langle f, \phi_n \rangle - c_n|^2}_{\geq 0}$$

$= 0$ iff $\langle f, \phi_n \rangle = c_n$ for all $n$.

Corollary: If $\{\phi_n\}_{n=1}^{\infty}$ orthonormal bases of $L^2(a,b)$. The partial sum $\sum_{n=1}^{N} \langle f, \phi_n \rangle \phi_n$ is the best approximation in norm of $f$ among all linear combinations of $\{\phi_1, \ldots, \phi_N\}$.

Problem 9 § 3.3 : Suppose $\{\phi_n\}$ is an orthornormal basis for $L^2(a,b)$ . Show that for any $f, g \in L^2(a,b)$

$$\langle f, g \rangle = \sum \langle f, \phi_n \rangle \overline{\langle g, \phi_n \rangle}$$

$$f = \sum_{n=1}^{\infty} \langle f, \phi_n \rangle \phi_n \qquad g = \sum_{m=1}^{\infty} \langle g, \phi_m \rangle \phi_m$$

$$\langle f, g \rangle = \left\langle \lim_{N\to\infty} \sum_{n=1}^{N} \langle f, \phi_n \rangle \phi_n \,,\, \lim_{M\to\infty} \sum_{m=1}^{M} \langle g, \phi_m \rangle \phi_m \right\rangle$$

continuity of inner product $\nearrow$

$$= \lim_{N\to\infty} \lim_{M\to\infty} \left\langle \sum_{n=1}^{N} \langle f, \phi_n \rangle \phi_n \,,\, \sum_{m=1}^{M} \langle g, \phi_m \rangle \phi_m \right\rangle$$

$$\sum_{n=1}^{N} \sum_{m=1}^{M} \langle \langle f, \phi_n \rangle \phi_n \,;\, \langle g, \phi_m \rangle \phi_m \rangle$$

$$= \lim_{N\to\infty} \lim_{M\to\infty} \sum_{n=1}^{N} \sum_{m=1}^{M} \langle f, \phi_n \rangle \langle g, \phi_m \rangle \langle \phi_n, \phi_m \rangle$$

$$= \lim_{N\to\infty} \lim_{M\to\infty} \sum_{n=1}^{N} \sum_{m=1}^{M} \langle f, \phi_n \rangle \overline{\langle g, \phi_n \rangle} \, \delta_{nm}$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} \langle f, \phi_n \rangle \overline{\langle g, \phi_n \rangle}$$

By Cauchy - Schwarz

$$\sum_{n=1}^{\infty} |\langle f, \phi_n \rangle| |\overline{\langle g, \phi_n \rangle}| \le \left( \sum_{n=1}^{\infty} |\langle f, \phi_n \rangle|^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} |\langle g, \phi_n \rangle|^2 \right)^{1/2}$$

absolutely convergent thus convergent.

§ 3.3 problem 6 $\phi_n = \sqrt{\frac{2}{\ell}} \sin\frac{(n-1/2)\pi x}{\ell}$ $n \geq 1$

$\{\phi_n\}$ is an orthonormal set in $L^2[0,\ell]$

$$\int_0^\ell \phi_n(x)\phi_m(x)\,dx = \frac{2}{\ell}\int_0^\ell \sin\frac{(n-1/2)\pi x}{\ell}\sin\frac{(m-1/2)\pi x}{\ell}\,dx$$

$$= \frac{2}{\ell}\int_0^\ell \sin\frac{(2n-1)\pi x}{2\ell}\sin\frac{(2m-1)\pi x}{2\ell}\,dx \qquad u = \frac{\pi x}{2\ell}, \quad du = \frac{\pi}{2\ell}dx$$

$$= \frac{4}{\pi}\int_0^{\pi/2} \sin(2n-1)u\sin(2m-1)u\,du$$

$$\sin a \sin b = \frac{1}{2}(\cos(a-b) - \cos(a+b))$$

$$= \frac{4}{\pi}\int_0^{\pi/2} \frac{1}{2}(\cos(2n-2m)u - \cos(2n+2m-2)u)\,du$$

$$= \frac{2}{\pi}\begin{cases} \pi/2 - \int_0^{\pi/2}\cos(4n-2)u\,du \quad (0) & \text{if } n=m \\ \frac{1}{2n-2m}\sin(2n-2m)u\Big|_0^{\pi/2} - \frac{1}{2n+2m-2}\sin(2n+2m-2)u\Big|_0^{\pi/2} \quad (0) & \end{cases}$$

$$\int_0^l \phi_n(x)\,\phi_m(x)\,dx = \delta_{nm}$$

a - Let $\psi_k(x) = \frac{1}{\sqrt{l}} \sin \frac{k\pi x}{2l}$. Show that $\{\psi_k\}$ is an orthonormal basis in $L^2(0,2l)$.

Let $f \in L^2(0,2l)$ then $\theta = \frac{\pi}{2l} x \in (0,\pi)$ and $g(\theta) = f(x) \in L^2(-\pi,\pi)$ because

$$\int_0^{\pi} |g(\theta)|^2\,d\theta = \int_0^{\pi} \left|f\left(\frac{2l\theta}{\pi}\right)\right|^2 d\theta = \int_0^{2l} |f(x)|^2 \frac{\pi}{2l}\,dx < \infty$$

$$x = \frac{2l}{\pi}\theta$$

$$dx = \frac{2l}{\pi}\,d\theta$$

then since

$\sqrt{\frac{2}{\pi}} \sin k\theta$ basis in $L^2(0,\pi)$

$$g(\theta) = \sum_{k=1}^{\infty} \left\langle g, \frac{\sqrt{2}}{\sqrt{\pi}} \sin k\theta \right\rangle \frac{\sqrt{2}}{\sqrt{\pi}} \sin k\theta$$

$$\left\langle g, \frac{\sqrt{2}}{\sqrt{\pi}} \sin k\theta \right\rangle = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{\pi} g(\theta) \sin k\theta\,d\theta = \frac{\sqrt{\pi}}{\sqrt{2l}} \int_0^{2l} f(x) \sin \frac{k\pi x}{2l}\,dx$$

$$\theta = \frac{\pi}{2l} x$$

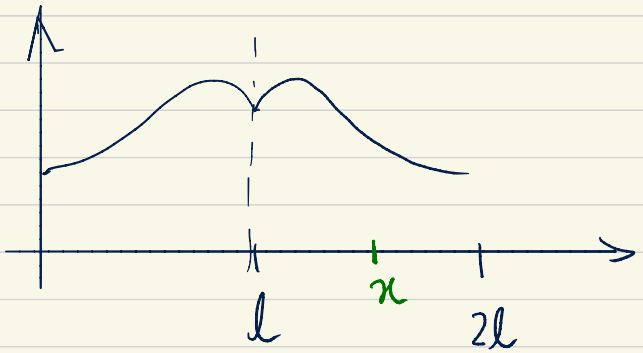
$$f(x) = \sum_{n=1}^{\infty} \frac{1}{\ell} \left\langle f, \sin\frac{k\pi x}{2\ell} \right\rangle \sin\frac{k\pi x}{2\ell} = \sum_{k=1}^{\infty} \langle f, \psi_k \rangle \psi_k$$

$$\int_0^{2\ell} \psi_k^2 \, dx = \frac{1}{\ell} \int_0^{2\ell} \sin^2 \frac{k\pi x}{2\ell} dx = \frac{2}{\pi} \int_0^{\pi} \sin^2 k\theta \, d\theta = 1$$

$\theta = \frac{\pi x}{2\ell}$ $\quad d\theta = \pi \frac{dx}{2\ell}$

Thus $\{\psi_k\}$ orthonormal basis for $L^2(0,2\ell)$

b- If $f \in L^2(0,\ell)$, extend $f$ to $[0,2\ell]$ by making it symmetric w.r.t. the line $x = \ell$




$$\tilde{f}(x) = \begin{cases} f(x) & 0 \le x \le \ell \\ f(2\ell - x) & \ell < x \le 2\ell \end{cases}$$

Show that

$$\langle \tilde{f}, \psi_{2n} \rangle = 0$$

and

$$\langle \tilde{f}, \psi_{2n-1} \rangle = \sqrt{2} \langle f, \phi_n \rangle$$

Recall $\psi_k(x) = \frac{1}{\sqrt{l}} \sin \frac{k\pi x}{2l}$.

$$\langle \tilde{f}, \psi_{2n} \rangle = \int_0^{2l} \tilde{f}(x) \frac{1}{\sqrt{l}} \sin \frac{2n\pi x}{2l}\, dx = \underbrace{\frac{1}{\sqrt{l}} \int_0^{l} f(x) \sin \frac{n\pi x}{l}\, dx}_{①}$$

$$+ \underbrace{\frac{1}{\sqrt{l}} \int_l^{2l} f(2l-x) \sin \frac{n\pi x}{l}\, dx}_{②}$$

$u = 2l - x$, $du = -dx$

$$\frac{1}{\sqrt{l}} \int_0^{l} f(u) \sin \frac{n\pi (2l-u)}{l}\, du$$

$$\sin \frac{n\pi(2l-u)}{l} = \sin\left(2\pi n - \frac{n\pi u}{l}\right)$$

$$= \sin 2\pi n \cos \frac{n\pi u}{l} - \underbrace{\cos 2\pi n}_{1} \sin \frac{n\pi u}{l}$$

$\sin 2\pi n = 0$

$$\underbrace{- \frac{1}{\sqrt{l}} \int_0^{l} f(u) \sin \frac{\pi n u}{l}\, du}_{-①}$$

$$\Rightarrow \quad \langle \tilde{f}, \psi_{2n} \rangle = 0.$$

$$\langle \tilde{f}, \psi_{2n-1} \rangle = \frac{1}{\sqrt{l}} \int_0^l f(x) \sin \frac{(2n-1)\pi x}{2l}\, dx + \frac{1}{\sqrt{l}} \int_l^{2l} f(2l-x) \sin \frac{(2n-1)\pi x}{2l}$$

$$\frac{1}{\sqrt{l}} \int_l^{2l} f(2l-x) \sin \frac{(2n-1)\pi x}{2l}\, dx = \frac{1}{\sqrt{l}} \int_0^l f(x) \sin \frac{(2n-1)\pi (2l-u)}{2l}\, du$$

$$u = 2l - x$$

$$du = -dx$$

$$\sin (2n-1)\pi \left(1 - \frac{u}{2l}\right) = \sin\left((2n-1)\pi - (2n-1)\pi \frac{u}{2l}\right) = -\cos(2n-1)\pi \sin (2n-1)\frac{\pi u}{2l}$$

$$= \sin (2n-1)\frac{\pi u}{2l}$$

$$\frac{1}{\sqrt{l}} \int_l^{2l} f(2l-x) \sin \frac{(2n-1)\pi x}{2l}\, dx = \frac{1}{\sqrt{l}} \int_0^l f(x) \sin \frac{(2n-1)\pi u}{2l}\, du \quad \text{thus}$$

$$\langle \tilde{f}, \psi_n \rangle = \frac{2}{\sqrt{l}} \int_0^l f(x) \sin \left(n - \frac{1}{2}\right) \frac{\pi x}{l}\, dx = \sqrt{2}\, \langle f, \phi_n \rangle$$

c. Conclude that if $f \in L^2(0,l)$ is such that $\langle f, \phi_n \rangle = 0$ then $f = 0$.

Extend $f$ to $\tilde{f}$ note that $\langle \tilde{f}, \psi_{2n} \rangle = 0$ and $\langle \tilde{f}, \psi_{2n-1} \rangle = \sqrt{2} \langle f, \phi_n \rangle = 0$. Thus $\langle \tilde{f}, \psi_k \rangle = 0$ for all $k$ - since $\{\psi_k\}$ orthonormal basis in $L^2(0,2l)$ then $\tilde{f} = 0 \Rightarrow f = 0$. Thus $\{\phi_n\}$ orthonormal basis for $L^2(0,l)$.