§ 2.5 problem 3

a) $$u(x,t) = \frac{400}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \exp\left(\frac{-(2n-1)^2 \pi^2 (1.1) t}{10^4} \right) \sin \frac{(2n-1)\pi x}{100}$$

b) $$u(50,t) = \frac{400}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \exp\left(\frac{-(2n-1)^2 \pi^2 (1.1) t}{10^4} \right)$$

since it is an alternating series with decreasing terms & 1st term ≥ 0

$$\frac{400}{\pi} \sum_{n=2m+1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \exp\left(\frac{-(2n-1)^2 \pi^2 (1.1) t}{10^4} \right) \geq 0$$

error when is $\frac{400}{\pi} \frac{1}{2n-1} \exp\left(\frac{-(2n-1)^2 \pi^2 (1.1) 30}{10^4} \right) \sim 1$

1.01

$\boxed{n = 6}$

$L^2(a,b) = \left\{ f : \int_a^b |f|^2 \, dx < \infty \right\}$ is the space of square integrable functions. Here we replace the Riemann integral for the Lebesgue integral.

Recall that $st \leq \frac{1}{2}(s^2+t^2)$ because $(t-s)^2 = t^2+s^2-2st \geq 0$

$\Rightarrow \quad 2st \leq s^2+t^2$

Let $f, g \in L^2(a,b)$ then $|f(x)\bar{g}(x)| \leq \frac{1}{2}(|f|^2+|g|^2)$ and the integral

$$\langle f, g \rangle = \int_a^b f(x)\,\bar{g}(x)\,dx \quad \text{is absolutely convergent.}$$

$\langle f, g \rangle$: inner product well defined in $L^2(a,b)$

If $\int_a^b |f|^2 \, dx = 0 \implies f = 0$ almost everywhere

($f \equiv 0$ except for a set of measure 0)

A subset $E \subset \mathbb{R}$ has measure 0 if for any $\varepsilon > 0$ E can be covered by a sequence of open intervals $\{I_j\}_j$ of length l_j such that $\sum_{j=1}^{\infty} l_j \leq \varepsilon$.

Examples: ① $\{r_j\}_j$ rationals in $[0,1]$ $\quad I_j = \left(r_j - \frac{\varepsilon}{2^j}, r_j + \frac{\varepsilon}{2^j}\right) \quad j \geq 1$.

② $1/3$ Cantor set. //

$$l_j = l(I_j) = \frac{2\varepsilon}{2^j} = \frac{\varepsilon}{2^{j-1}} \Rightarrow \sum_{j=1}^{\infty} l_j = \varepsilon$$

Theorem: ① $L^2(a,b)$ is complete with respect to the norm.

② For $f \in L^2(a,b)$ there is a sequence f_n of smooth functions in $[a,b]$ such that $f_n \longrightarrow f$ in norm

1/3 Cantor set



$$C = \bigcap_{n=0}^{\infty} C_n$$

↑

C is a set of measure 0

it is uncountable

Remarks : ① Relationship between $L^2(a,b)$ & $PC(a,b)$

- $PC(a,b)$ are dense in $L^2(a,b)$ with $\| \ \|$

② Can we find FS expansions of functions in $L^2(a,b)$?

Recall Bessel's inequality: If f is 2π periodic and Riemann integrable on $[-\pi, \pi]$ with Fourier coefficients c_n then

$$\sum_{n=-\infty}^{\infty} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 \, d\theta$$

Bessel's inequality: If $\{\phi_n\}_{n=1}^{\infty}$ orthonormal set in $L^2(a,b)$ and $f \in L^2(a,b)$ then

$$\sum_{n=1}^{\infty} |\langle f, \phi_n \rangle|^2 \leq \|f\|^2$$

Proof: By the Pythagorean theorem

$$\left\| \sum_{n=1}^{N} \langle f, \phi_n \rangle \phi_n \right\|^2 = \sum_{n=1}^{N} |\langle f, \phi_n \rangle|^2 \|\phi_n\|^2 = \sum_{n=1}^{N} |\langle f, \phi_n \rangle|^2$$

$$0 \leq \left\| f - \sum_{n=1}^{N} \langle f, \phi_n \rangle \phi_n \right\|^2 = \|f\|^2 - 2\mathrm{Re} \left\langle f, \sum_{n=1}^{N} \langle f, \phi_n \rangle \phi_n \right\rangle + \left\| \sum_{n=1}^{N} \langle f, \phi_n \rangle \phi_n \right\|^2$$

$$= \|f\|^2 - 2\mathrm{Re} \sum_{n=1}^{N} \overline{\langle f, \phi_n \rangle} \langle f, \phi_n \rangle + \sum_{n=1}^{N} |\langle f, \phi_n \rangle|^2$$

$$0 \le \|f\|^2 - 2\mathrm{Re} \sum_{n=1}^{N} \overline{\langle f, \phi_n \rangle} \langle f, \phi_n \rangle + \sum_{n=1}^{N} |\langle f, \phi_n \rangle|^2$$

$$0 \le \|f\|^2 - \sum_{n=1}^{N} |\langle f, \phi_n \rangle|^2 \quad \Rightarrow \quad \sum_{n=1}^{N} |\langle f, \phi_n \rangle|^2 \le \|f\|^2$$

$N \longrightarrow \infty$

Q: Given an orthonormal set $\{\phi_n\}_{n=1}^{\infty}$ in $L^2(a,b)$ is it true that

$$f = \sum_{n=1}^{\infty} \langle f, \phi_n \rangle \phi_n \; ?$$

Issues? ① Convergence, in what sense is there convergence?

② Are there enough $\{\phi_n\}$?

Lemma: If $f \in L^2(a,b)$ and $\{\phi_n\}_{n=1}^{\infty}$ is an orthonormal set in $L^2(a,b)$ then the series $\sum_{n=1}^{\infty} \langle f, \phi_n \rangle \phi_n$ converges in norm and

$$\left\| \sum_{n=1}^{\infty} \langle f, \phi_n \rangle \phi_n \right\| \leq \|f\| .$$

Proof: Bessel's inequality guarantees that the series $\sum_{n=1}^{\infty} |\langle f, \phi_n \rangle|^2$ converges. If $m < n$

$$\left\| \sum_{l=1}^{n} \langle f, \phi_l \rangle \phi_l - \sum_{l=1}^{m} \langle f, \phi_l \rangle \phi_l \right\|^2 = \left\| \sum_{l=m+1}^{n} \langle f, \phi_l \rangle \phi_l \right\|^2$$

$$= \sum_{l=m+1}^{n} |\langle f, \phi_l \rangle|^2 \longrightarrow 0 \quad (n, m \to \infty)$$

↑ Pythagorean theorem

Thus $S_n(f) = \sum_{l=1}^{n} \langle f, \phi_l \rangle \phi_l$ is a Cauchy sequence in $L^2(a,b)$. Since $L^2(a,b)$ is complete the series converges. Thus

$$\left\| \sum_{n=1}^{\infty} \langle f, \phi_n \rangle \phi_n \right\|^2 = \left\| \lim_{N \to \infty} \sum_{n=1}^{N} \langle f, \phi_n \rangle \phi_n \right\|^2 =$$

$$= \lim_{N\to\infty} \left\| \sum_{n=1}^{N} \langle f, \phi_n \rangle \phi_n \right\|^2 = \lim_{N\to\infty} \sum_{n=1}^{N} |\langle f, \phi_n \rangle|^2 = \sum_{n=1}^{\infty} |\langle f, \phi_n \rangle|^2$$

↑ continuity of the norm

↑ Pythagorean theorem

$\leq \|f\|^2$ ← Bessel's inequality

$$\left\| \sum_{n=1}^{\infty} \langle f, \phi_n \rangle \phi_n \right\|^2 \leq \|f\|^2$$

Q: How do we know there are "enough" ϕ_n's ?

Theorem: Let $\{\phi_n\}_{n=1}^{\infty}$ be an orthonormal set in $L^2(a,b)$. The following conditions are equivalent:

(1) If $\langle f, \phi_n \rangle = 0$ for all n then $f = 0$.

(2) For every $f \in L^2(a,b)$ we have $f = \sum_{n=1}^{\infty} \langle f, \phi_n \rangle \phi_n$ where the series converges in norm.

(3) For every $f \in L^2(a,b)$, we have Parseval's equation

$$\|f\|^2 = \sum_{n=1}^{\infty} |\langle f, \phi_n \rangle|^2$$

Such an orthonormal set is an orthonormal basis.

Proof: (1) $\Rightarrow$ (2); (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1)

(1) $\Rightarrow$ (2) $f \in L^2(a,b)$ then $\sum_{n=1}^{\infty} \langle f, \phi_n \rangle \phi_n$ converges in norm then

$$g = f - \sum_{n=1}^{\infty} \langle f, \phi_n \rangle \phi_n \in L^2(a,b) \quad \text{and}$$

$$\langle g, \phi_m \rangle \overset{(1)}{=} \langle f, \phi_m \rangle - \sum_{n=1}^{\infty} \langle f, \phi_n \rangle \underbrace{\langle \phi_n, \phi_m \rangle}_{\delta_{nm}} = \langle f, \phi_m \rangle - \langle f, \phi_m \rangle = 0$$

$\Rightarrow g = 0$

② $\Rightarrow$ ③ the Pythagorean theorem and the calculation we did before

$$\|f\|_2^2 = \left\| \sum_{n=1}^{\infty} \langle f, \phi_n \rangle \phi_n \right\|^2 \leq \sum_{n=1}^{\infty} |\langle f, \phi_n \rangle|^2$$

③ $\Rightarrow$ ① if $\langle f, \phi_n \rangle = 0$ for all n by ③ $\|f\| = 0 \Rightarrow f \equiv 0$.

Theorem: $\left\{ \frac{1}{\sqrt{2\pi}} e^{inx} \right\}_{n=-\infty}^{\infty}$ or $\left\{ \frac{1}{\sqrt{\pi}} \cos nx \right\}_{n=1}^{\infty} \cup \left\{ \frac{1}{\sqrt{\pi}} \sin nx \right\}_{n=1}^{\infty} \cup \left\{ \frac{1}{\sqrt{2\pi}} \right\}$

are orthonormal basis for $L^2(-\pi, \pi)$.

Thus

$$\int_{-\pi}^{\pi} |f|^2 dx = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} |\langle f, e^{inx} \rangle|^2 = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} f(x) e^{-inx} dx \right|^2$$

$$= \sum_{n=-\infty}^{\infty} 2\pi \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \right|^2$$

$$= 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2$$

(equality in Bessel's ineq if $f \in PC(a,b)$)

(1) Find an orthogonal (orthonormal) basis for $L^2(0,\pi)$.

(2) What do we need to prove the previous theorem?

- approximate by piecewise continuous functions

Let f be a periodic function, then the Fourier series of f converges to f

(i) absolutely, uniformly and in norm if f continuous and piecewise smooth

(ii) pointwise and in norm if f piecewise smooth (modulo the choice of the right representative)

(iii) in norm if $f \in L^2(a,b)$

Problem 10c. § 3.3 Evaluate $\sum_{n=1}^{\infty} \frac{n^2}{(n^2+1)^2}$ applying Parseval's equation to one of the Fourier expansion in Table 1 § 2.1

Problem 6 § 3.4 Find an example of a sequence $\{f_n\}$ in $L^2(0,\infty)$ such that $f_n \to 0$ uniformly but $f_n \not\to 0$ in norm