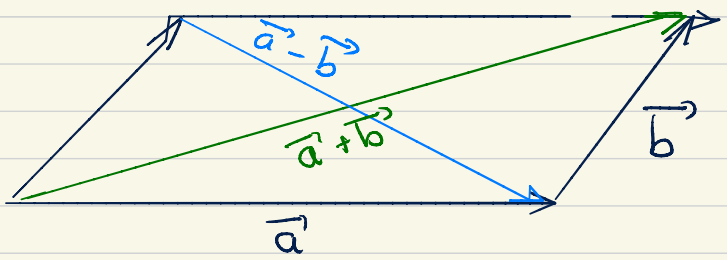
§ 3.1 Problem 1. Show that for all $\vec{a}, \vec{b} \in \mathbb{C}^k$

$$\|\vec{a}+\vec{b}\|^2 + \|\vec{a}-\vec{b}\|^2 = 2(\|\vec{a}\|^2 + \|\vec{b}\|^2)$$



① $\|\vec{a}+\vec{b}\|^2 = \|\vec{a}\|^2 + 2\mathrm{Re}\langle\vec{a},\vec{b}\rangle + \|\vec{b}\|^2$

$\|\vec{a}-\vec{b}\|^2 = \|\vec{a}\|^2 + 2\mathrm{Re}\langle\vec{a},-\vec{b}\rangle + \|\vec{b}\|^2$

② $= \|\vec{a}\|^2 - 2\mathrm{Re}\langle\vec{a},\vec{b}\rangle + \|\vec{b}\|^2$

① + ② $= \|\vec{a}+\vec{b}\|^2 + \|\vec{a}-\vec{b}\|^2 = 2(\|\vec{a}\|^2 + \|\vec{b}\|^2)$

§ 3.1 Problem 2 Suppose $\{\vec{y}_1 \dots \vec{y}_k\}$ orthogonal set in $\mathbb{C}^k$

Show that for any $\vec{a} \in \mathbb{C}^k$

$$\vec{a} = \frac{\langle a, \vec{y}_1 \rangle}{\|\vec{y}_1\|} \frac{\vec{y}_1}{\|\vec{y}_1\|} + \dots + \frac{\langle a, \vec{y}_k \rangle}{\|\vec{y}_k\|} \frac{\vec{y}_k}{\|\vec{y}_k\|^2}$$

$\vec{u}_j = \dfrac{\vec{y}_j}{\|\vec{y}_j\|}$ $\{u_1 \dots u_k\}$ orthonormal basis + Theorem 3.1

$$\vec{a} = \langle \vec{a}, \vec{u}_1 \rangle \vec{u}_1 + \langle \vec{a}, \vec{u}_2 \rangle \vec{u}_2 + \dots + \langle \vec{a}, \vec{u}_k \rangle \vec{u}_k$$

Recall that the proofs of Cauchy - Schwarz, the Triangle Inequality and the Pythagorean theorem only depend on the fact that $\mathbb{C}^k$ is a linear (vector) space and that the inner product satisfies

i) $\langle \vec{a} + \vec{b}, \vec{c} \rangle = \langle \vec{a}, \vec{c} \rangle + \langle \vec{b}, \vec{c} \rangle$

iii) $\langle \alpha \vec{a}, \vec{b} \rangle = \alpha \langle \vec{a}, \vec{b} \rangle$

v) $\langle \vec{b}, \vec{a} \rangle = \overline{\langle \vec{a}, \vec{b} \rangle}$

and the definition of norm in terms of inner product

$$\|\vec{a}\| = \sqrt{\langle \vec{a}, \vec{a} \rangle}$$

Cauchy - Schwarz $\quad |\langle \vec{a}, \vec{b} \rangle| \leq \|\vec{a}\| \, \|\vec{b}\|$

Triangle inequality $\quad \|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$

Functions & inner products

Let

$$PC(a,b) = \{ f : (a,b) \longrightarrow \mathbb{C} : \text{piece wise continuous} \}$$

$\uparrow$ linear space if $f, g \in PC(a,b)$ and $\alpha, \beta \in \mathbb{C}$ then $\alpha f + \beta g \in PC(a,b)$

Define $\quad \langle f, g \rangle = \int_a^b f(x)\,\bar{g}(x)\,dx$

(i) $\langle f+g, h \rangle = \int_a^b (f(x)+g(x))\,\bar{h}(x)\,dx = \underbrace{\int_a^b f(x)\bar{h}(x)\,dx}_{\langle f,h \rangle} + \underbrace{\int_a^b g(x)\,\bar{h}(x)\,dx}_{\langle g,h \rangle}$

(iii) $\langle \alpha f, g \rangle = \int_a^b \alpha f(x)\,\bar{g}(x)\,dx = \alpha \int_a^b f(x)\bar{g}(x)\,dx = \alpha \langle f, g \rangle$

(v) $\langle g, f \rangle = \int_a^b g(x)\,\overline{f(x)}\,dx \underset{\uparrow}{=} \int_a^b \overline{f(x)\,\overline{g(x)}}\,dx \underset{\uparrow}{=} \overline{\int_a^b f(x)\,\overline{g(x)}\,dx}$

$= \overline{\langle f, g \rangle}$

(check) $a\bar{b} = \overline{\bar{a}b}$

(check) $\int_a^b \bar{h}(x)\,dx = \overline{\int_a^b h(x)\,dx}$

$$\langle f, g \rangle = \int_a^b f \bar{g}\, dx \qquad \|f\| = \left(\int_a^b f \bar{f}\, dx \right)^{1/2} = \left(\int_a^b |f|^2 dx \right)^{1/2}$$

Cauchy - Schwarz $|\langle f, g \rangle| \leq \|f\| \|g\|$ $\leftarrow$

$$\left| \int_a^b f(x) \bar{g}(x)\, dx \right| \leq \left(\int_a^b |f|^2 dx \right)^{1/2} \left(\int_a^b |g|^2 dx \right)^{1/2}$$

Triangle inequality $\|f + g\| \leq \|f\| + \|g\|$

$$\left(\int_a^b |f+g|^2 dx \right)^{1/2} \leq \left(\int_a^b |f|^2 dx \right)^{1/2} + \left(\int_a^b |g|^2 dx \right)^{1/2}$$

Define for $f \in PC(a,b)$ $\quad \|f\| = \left(\int_a^b |f(x)|^2 \, dx \right)^{1/2}$.

Hence Cauchy - Schwarz, triangle inequality and Pythagorean theorem hold.

(CS) $$\left| \int_a^b f(x)\overline{g}(x)\,dx \right| \leq \left(\int_a^b |f(x)|^2 \, dx \right)^{1/2} \left(\int_a^b |g(x)|^2 \, dx \right)^{1/2}$$

($\Delta\leq$) $$\left(\int_a^b |f(x)+g(x)|^2 \, dx \right)^{1/2} \leq \left(\int_a^b |f(x)|^2 \, dx \right)^{1/2} + \left(\int_a^b |g(x)|^2 \, dx \right)^{1/2}$$

(PT) $$\int_a^b \left| \sum_{j=1}^n f_j(x) \right|^2 dx = \sum_{j=1}^n \int_a^b |f_j(x)|^2 \, dx$$ provided

$$\int_a^b f_j(x)\overline{f_l}(x)\,dx = 0 \qquad \text{for } j \neq l$$

Recall that if $f \in PC(a,b)$ we assume $f(x) = \frac{1}{2}(f(x+) + f(x-))$ for $x \in (a,b)$; $f(a) = f(a+)$ & $f(b) = f(b-)$

In homework you show that if $\tilde{f} \in PC(a,b)$ $\tilde{f} \geq 0$ and $\int_a^b \tilde{f}(x)\,dx = 0$ then $\tilde{f}(x) = 0$ (at all but finitely many points is not using convention above) thus

$\|f\| = 0 \Rightarrow \int_a^b |f|^2 dx = 0$ iff $f = 0$ (except possibly at finitely many points)

Question: Given an orthonormal set $\{\phi_n\}_n \subset PC(a,b)$ can we express $f \in PC(a,b)$ as $\sum \langle f, \phi_n \rangle \phi_n$?

What is the difference between $\mathbb{C}^k$ and $PC(a,b)$?

- $\mathbb{C}^k$ finite dimensional vs $PC(a,b)$ infinite dimensional
- convergence of series an issue.

Consider $\phi_n = \frac{1}{\sqrt{2\pi}} e^{inx}$ $n = 0, \pm 1, \pm 2, \dots$ in $PC(-\pi, \pi)$.

$$\langle \phi_m, \phi_n \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{imx} \overline{e^{inx}}\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{imx} e^{-inx}\, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(m-n)x}\, dx = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

$\Rightarrow \{\phi_n\}$ orthonormal set. - Moreover if $f \in PC(-\pi, \pi)$ with Fourier coefficients

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{e^{inx}}\, dx = \frac{1}{\sqrt{2\pi}} \langle f, \phi_n \rangle$$

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \langle f, \phi_n \rangle \sqrt{2\pi}\, \phi_n = \sum_{n=-\infty}^{\infty} \langle f, \phi_n \rangle \phi_n \quad (*)$$

modulo convergence of the Fourier series we have that f can be expressed as in $(*)$.

Consider $\{\cos nx\}_{n=0}^{\infty}$ on $[0,\pi]$ then since

$$\cos a \cos b = \frac{1}{2}\left[\cos(a+b) + \cos(a-b)\right]$$ we have

$$\int_0^{\pi} \cos nx \cos mx \, dx = \frac{1}{2}\int_0^{\pi}\left[\cos(n+m)x + \cos(n-m)x\right]dx \quad *$$

in general

$$\int_0^{\pi} \cos kx \, dx = \begin{cases} \int_0^{\pi} dx = \pi & \text{if } k=0 \\ \frac{1}{k}\sin kx\Big|_0^{\pi} = 0 & \text{if } k\neq 0 \end{cases}$$

then

$$\int_0^{\pi} \cos nx \cos mx \, dx = \begin{cases} \pi & n=m=0 \\ \pi/2 & n=m\neq 0 \\ 0 & n\neq m \\ & n\neq 0 \ \ m\neq 0 \end{cases}$$

$$\int_0^{\pi} \cos nx \cos mx \, dx = \begin{cases} \pi & \text{if } n = m = 0 \\ \pi/2 & \text{if } n = m \neq 0 \\ 0 & \text{if } n \neq m \end{cases}$$

let $\psi_0(x) = \left(\frac{1}{\pi}\right)^{1/2}$ & $\psi_n(x) = \frac{2}{\sqrt{\pi}} \cos nx$, $\{\psi_n\}_{n=0}^{\infty}$ orthonormal set in $PC(0,\pi)$. The Fourier cosine coefficient a_n is

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx = \begin{cases} 2(1/\pi)^{1/2} \langle f, \psi_0 \rangle & n = 0 \\ (2/\pi)^{1/2} \langle f, \psi_n \rangle & n > 0 \end{cases}$$

hence

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx = \sum_{n=0}^{\infty} \langle f, \psi_n \rangle \psi_n(x)$$

Once again the question of representation as above is a question of convergence.

Questions: ① Are there families of orthonormal sets such that all functions in $PC(a,b)$ can be written as linear combinations of its elements?

② How many elements do these sets need to have?

③ Examples?

Convergence & Completness

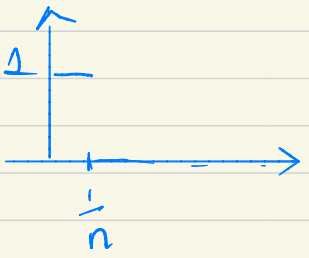
Recall that in $\mathbb{C}^k$, there is a distance associated with the norm $\|\vec{a} - \vec{b}\|$ = distance between $\vec{a}$ & $\vec{b}$. We say $\vec{a_n} \to \vec{a}$ as $n \to \infty$ if $\|\vec{a_n} - \vec{a}\| \to 0$ as $n \to \infty$

If $\{f_n\}$ is a sequence in $PC(a,b)$ we say that $f_n \to f$ in norm (in L^2) if $\|f_n - f\| \to 0$ as $n \to \infty$; i.e.

$$f_n \to f \quad \text{if} \quad \int_a^b |f_n(x) - f(x)|^2 \, dx \to 0 \quad (n \to \infty)$$

Examples: If $[a,b] = [0,1]$

1) $$f_n(x) = \begin{cases} 1 & 0 \le x \le 1/n \\ 0 & \frac{1}{n} < x < 1 \end{cases}$$

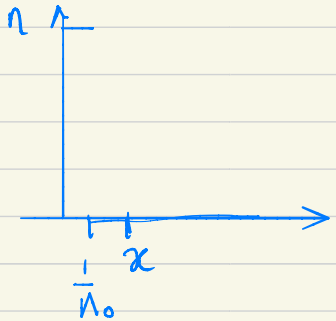


$$\|f_n\| = \left(\int_0^1 f_n^2 \, dx \right)^{1/2} = \left(\int_0^{1/n} dx \right)^{1/2} = \left(\frac{1}{n} \right)^{1/2} \to 0 \quad (n \to \infty)$$

& $f_n(0) = 1$ for all n

thus $f_n \longrightarrow 0$ in norm but $f_n \not\to 0$ pointwise

2) $$g_n(x) = \begin{cases} n & \text{for } 0 < x < 1/n \\ 0 & \text{elsewhere} \end{cases}$$



$$\|g_n\| = \left(\int_0^1 n^2 \, dx \right)^{1/2} = n \left(\int_0^{1/n} dx \right)^{1/2} = n \frac{1}{\sqrt{n}} = \sqrt{n} \longrightarrow \infty$$
$$n \to \infty$$

On the other hand $g_n(0) = 0$ for all n, $\forall\, x > 0$
$\exists\, n_0$ such that $x > \frac{1}{n_0}$ and for $n \geq n_0$ $g_n(x) = 0$

hence $g_n \longrightarrow 0$ pointwise but g_n does not converge in norm.

Q: What is the relationship between pointwise convergence and norm convergence?

Theorem: If $f_n \longrightarrow f$ uniformly on $[a,b]$ $(-\infty < a < b < \infty)$ then $f_n \longrightarrow f$ in norm.

$f_n \longrightarrow f$ uniformly on $[a,b]$ means that there is a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ such that for all $x \in [a,b]$

$$|f_n(x) - f(x)| < \varepsilon_n \quad \text{and} \quad \varepsilon_n \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Proof

$$\|f_n - f\|^2 = \int_a^b |f_n(x) - f(x)|^2 \, dx \underset{\substack{\uparrow \\ \text{by uniform} \\ \text{convergence}}}{\leq} \int_a^b \varepsilon_n^2 \, dx = \underbrace{\varepsilon_n^2 (b-a)}_{\substack{\downarrow \, n \to 0 \\ 0}}.$$

Properties: If $\|f_n - f\| \to 0$ then

① $\|f_n\| \longrightarrow \|f\|$

② $\langle f_n, g \rangle \longrightarrow \langle f, g \rangle$ and ③ $\langle g, f_n \rangle \longrightarrow \langle g, f \rangle$

$\|f_n - f\| \to 0 \quad n \to \infty$ ① $\|f_n\| \to \|f\|$

② $\langle f_n, g \rangle \to \langle f, g \rangle$

① $\boxed{\|f_n - f\| \geq \|f_n\| - \|f\|}$ $\left(\|f_n\| \leq \|f_n - f\| + \|f\|\right)$

$\downarrow n \to \infty$

$0 \geq \limsup_{n\to\infty} \|f_n\| - \|f\| \Rightarrow \limsup_{n\to\infty} \|f_n\| \leq \|f\|$

$\boxed{\|f_n - f\| \geq \|f\| - \|f_n\|}$

$\to \|f_n - f\| \geq \big|\|f_n\| - \|f\|\big| \geq 0$

$n \to \infty \downarrow$

$0 \quad : \quad \lim_{n\to\infty} \big|\|f_n\| - \|f\|\big| = 0$ squeeze theorem

② $|\langle f_n, g\rangle - \langle f, g\rangle| \overset{\text{linearity}}{=} |\langle f_n - f, g\rangle| \overset{\text{CS}}{\leq} \|f_n - f\|\,\|g\|$

$\downarrow n \to \infty$

0

$PC(a,b)$ is not <u>complete</u>, in the same way that the rationals are not complete: there are sequences of rationals that do not converge to a rational but converge to a real number.

$\{\vec{a}_n\}_{n=1}^{\infty}$ is a Cauchy sequence if $\|\vec{a}_n - \vec{a}_m\| \xrightarrow[n,m\to\infty]{} 0$

A space S of vectors is complete if every Cauchy sequence in S has a limit in S.

Examples: ① $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^k$ and $\mathbb{C}^k$ are complete $\mathbb{Q}$ is not complete

② $[a,b] = [0,1]$

$$f_n(x) = \begin{cases} x^{-1/4} & x > 1/n \\ 0 & 0 \le x \le 1/n \end{cases}$$

$m > n$

$$f_m(x) - f_n(x) = \begin{cases} x^{-1/4} & 1/m < x \le 1/n \\ 0 & \text{otherwise} \end{cases}$$

$$\|f_m - f_n\|^2 = \int_{1/m}^{1/n} (x^{-1/4})^2 \, dx = \int_{1/m}^{1/n} x^{-1/2} \, dx = 2x^{1/2} \Big|_{1/m}^{1/n}$$

$$\|f_m - f_n\|^2 = 2\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{m}}\right) \longrightarrow 0 \quad (n, m \to \infty)$$

$\{f_n\}$ is Cauchy. Note that $f_n \xrightarrow{\text{pointwise}} f = \begin{cases} x^{-1/4} & x > 0 \\ 0 & x = 0 \end{cases}$

$$\int_0^1 (f_n - f)^2 \, dx = \int_0^{1/n} x^{-1/2} \, dx = 2x^{1/2} \Big|_0^{1/n} = 2\frac{1}{\sqrt{n}} \longrightarrow 0$$

and $f_n \to f$ in norm but f is not bounded does not piecewise continuous.

The same way the rationals were enlarged to the reals to make the set complete we can enlarge $PC(a,b)$ with the norm $\| \ \|$ to make a complete space - We need to change

the Riemann integral for the Lebesgue integral.

$$L^2(a,b) = \left\{ f : \int_a^b |f|^2 \, dx < \infty \right\}$$

Recall that $st \leq \frac{1}{2}(s^2 + t^2)$ because

Let $f, g \in L^2(a,b)$ then $|f(x)\bar{g}(x)| \leq \frac{1}{2}(|f|^2 + |g|^2)$ and the integral

$$\langle f, g \rangle = \int_a^b f(x)\,\bar{g}(x)\,dx$$ is absolutely convergent.

↑ inner product

well defined in $L^2(a,b)$