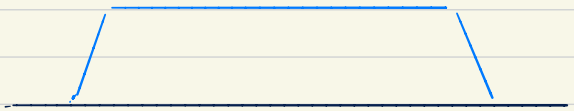
§ 2.5 problem 4 : Consider a vibrating string occupying the interval $0 \le x \le l$ - Suppose that the string is plucked in the middle in such a way that the initial displacement $u(x,0)$ is

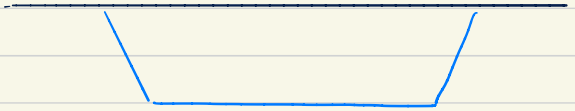
$$u(x,0) = \begin{cases} 2mx/l & \text{for } 0 \le x \le l/2 \\ 2m(l-x)/l & \text{for } l/2 \le x \le l \end{cases} \quad [f(x)]$$

and the initial velocity is $u_t(x,0)$ is 0. $[g(x) = 0]$

a- Find the displacement $u(x,t)$ as a Fourier series

b- Describe $u(x,t)$ using d'Alembert's formula - Show that for times $t > 0$, $u(x,t)$ (as a function of x) typically looks like





Find the Fourier series for f and g. Since the string is attached

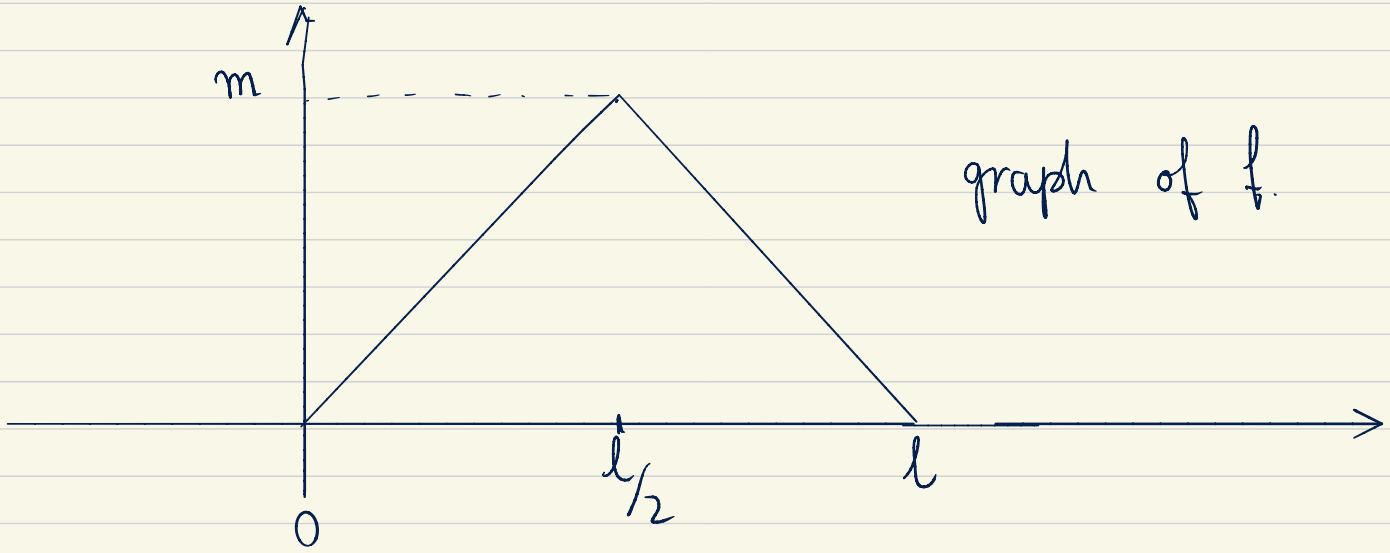
$u(0,t) = u(l,t) = 0$ (no displacement)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \qquad g(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}$$

$B_n = 0$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi c t}{l}$$

where $u_{tt} = c^2 u_{xx}$



graph of f.

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^{l/2}\frac{2mx}{l}\sin\frac{n\pi x}{l}\,dx$$

$$+\frac{2}{l}\int_{l/2}^{l}2m\left(\frac{l-x}{l}\right)\sin\frac{n\pi x}{l}\,dx$$

$$y = l - x \qquad dy = -dx$$

$$= \frac{4m}{l^2}\int_0^{l/2} x\sin\frac{n\pi x}{l}\,dx + \frac{4m}{l^2}\int_0^{l/2} y\sin\frac{n\pi(l-y)}{l}\,dy$$

$$\sin\left(n\pi - \frac{n\pi y}{l}\right)$$

$$\sin n\pi\cos\frac{n\pi y}{l} - \cos n\pi\sin\frac{n\pi y}{l}$$

$$0 \qquad -(-1)^n\sin\frac{n\pi y}{l}$$

$$= \frac{4m}{l^2}\left(1-(-1)^n\right)\int_0^l x\sin\frac{n\pi x}{l}\,dx$$

$$= \begin{cases} 0 & \text{if } n \text{ even} \\ \dfrac{8m}{l^2}\displaystyle\int_0^l x\sin\frac{n\pi x}{l}\,dx & \text{if } n \text{ odd} \end{cases}$$

If n is odd

$$\frac{8m}{l^2}\int_0^{l/2} x\sin\frac{n\pi x}{l}\,dx = \frac{8m}{l^2}\; x\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right)\Big|_0^{l/2} + \frac{8m}{ln\pi}\int_0^{l/2}\cos\frac{n\pi x}{l}\,dx$$

$\uparrow$ integration by parts

$$= \frac{8m}{l^2}\;\frac{l}{2}\left(-\frac{l}{n\pi}\right)\cos\frac{n\pi x}{2} + \frac{8m}{ln\pi}\;\frac{l}{n\pi}\sin\frac{n\pi x}{l}\Big|_0^{l/2}$$

0 (for n odd)

$$= \frac{8m}{n^2\pi^2}\sin\frac{n\pi}{2} = \frac{8}{(2k+1)^2\pi^2}\sin\left(k\pi+\frac{\pi}{2}\right)$$

$n = 2k+1$

$$= \frac{8m}{(2k+1)^2\pi^2}\cos k\pi\,\sin\pi/2 = \boxed{\frac{8(-1)^k}{\pi^2(2k+1)^2}m = b_{2k+1}}$$

$$\boxed{b_{2k}=0}$$

$$u(x,t) = \frac{8m}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin\frac{(2n+1)\pi x}{l} \cos\frac{(2n+1)\pi c t}{l} .$$

is a solution of $u_{tt} = c^2 u_{xx}$ $u(x,0) = f(x)$ $u_x(x,0) = 0$

$u(0,t) = u(l,t) = 0$

Contrast with solution to the heat equation $u_t = k u_{xx}$

with $u(x,0) = f(x)$, $u(0,t) = u(l,t) = 0$

$$u(x,t) = \frac{8m}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \exp\left(-\frac{(2n+1)^2 \pi^2 c t}{l^2}\right) \sin\frac{(2n+1)\pi x}{l}$$

b. ⊛ $u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2}[G(x+ct) - G(x-ct)]$

where $G' = g = 0 \Rightarrow$ G constant. (0)

$$f(x) = \begin{cases} 2mx/l & 0 \le x \le l/2 \\ 2m(l-x)/l & l/2 \le x \le l \end{cases}$$ (periodic extension)

D' alembert.

Vectors & inner products

Recall in $\mathbb{R}^3$, $\vec{a} = (a_1, a_2, a_3)$. The inner product (or dot) product of 2 vectors is $\langle \vec{a}, \vec{b} \rangle = \vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

The norm of a vector is defined by $\|\vec{a}\| = \sqrt{\langle \vec{a}, \vec{a} \rangle} = \sqrt{a_1^2 + a_2^2 + a_3^2}$

A complex k-dimensional vector is an ordered k-tuple of complex numbers $\vec{a} = (a_1, a_2, \dots, a_k)$, the space is denoted by $\mathbb{C}^k$

(1) $\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, \dots, a_k + b_k)$

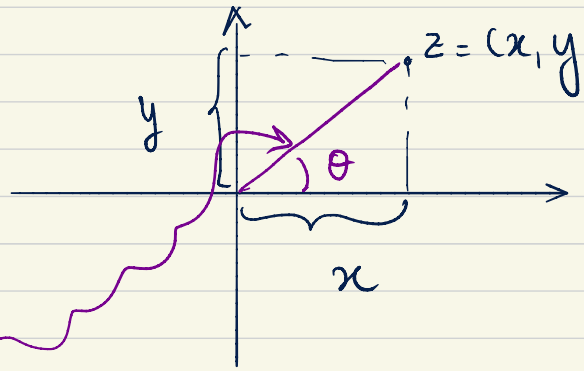
(2) $c \in \mathbb{C}$ $\quad c\vec{a} = (ca_1, ca_2, \dots, ca_k)$

(3) If $z = x + iy \in \mathbb{C}$, with $x, y \in \mathbb{R}$ then $\bar{z} = x - iy$

Note $z \in \mathbb{R}$ if and only if $z = \bar{z}$ because ...

Moreover $|z| = \sqrt{z\bar{z}} =$

$\mathbb{C} \sim \mathbb{R}^2$



$x, y \in \mathbb{R}$ real

$\bar{z} = x - iy$

$z \in \mathbb{R}$ iff $y = 0$

(iff = if and only if)

$z = x + iy$ ← imaginary part of z

Real part of z

$\underline{|z|} = \sqrt{z\bar{z}} = \sqrt{(x+iy)(x-iy)}$

$= \sqrt{x^2 + ixy - ixy - (iy)^2}$

$= \sqrt{x^2 + y^2}$

$\mathrm{Im}(z) = \frac{z - \bar{z}}{2i}$

$\mathrm{Re}(z) = \frac{z + \bar{z}}{2}$

$z = \sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}} + i\frac{y}{\sqrt{x^2+y^2}}\right) = \sqrt{x^2+y^2}(\cos\theta + i\sin\theta)$

$\cos\theta$ $\quad$ $\sin\theta$

$= \underline{|z|e^{i\theta} = z}$

(4) $\langle \vec{a}, \vec{b} \rangle = a_1 \overline{b_1} + a_2 \overline{b_2} + \cdots + a_k \overline{b_k}$

(5) $\|\vec{a}\| = \sqrt{\langle \vec{a}, \vec{a} \rangle} = (a_1 \overline{a_1} + \cdots a_k \overline{a_k})^{1/2} = (|a_1|^2 + \cdots + |a_k|^2)^{1/2}$

Properties : assume $\alpha, \beta \in \mathbb{C}$ & $\vec{a}, \vec{b}, \vec{c} \in \mathbb{C}^k$ then

* i) $\langle \vec{a} + \vec{b}, \vec{c} \rangle = \langle \vec{a}, \vec{c} \rangle + \langle \vec{b}, \vec{c} \rangle$

ii) $\langle \vec{a}, \vec{b} + \vec{c} \rangle = \langle \vec{a}, \vec{b} \rangle + \langle \vec{a}, \vec{c} \rangle$

* iii) $\langle \alpha \vec{a}, \vec{b} \rangle = \alpha \langle \vec{a}, \vec{b} \rangle$

iv) $\langle \vec{a}, \beta \vec{b} \rangle = \overline{\beta} \langle \vec{a}, \vec{b} \rangle$

* v) $\langle \vec{b}, \vec{a} \rangle = \overline{\langle \vec{a}, \vec{b} \rangle}$ inner product hermitian

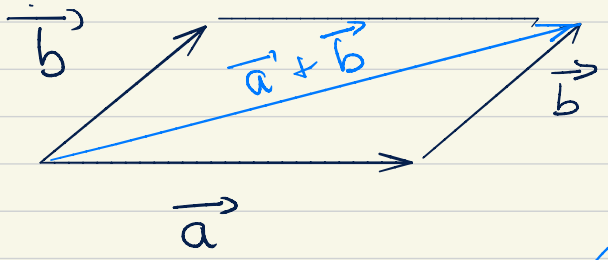
(vi) $\|\alpha \vec{a}\| = |\alpha| \|\vec{a}\|$

(vii) $\|\vec{a}\| > 0$ for all $\vec{a} \neq \vec{0} = (0, \ldots 0)$

Lemma : For $\vec{a}$ and $\vec{b}$ in $\mathbb{C}^k$

$$\|\vec{a}+\vec{b}\|^2 = \|\vec{a}\|^2 + 2\,\mathrm{Re}\langle \vec{a},\vec{b}\rangle + \|\vec{b}\|^2$$

Proof



$$\|\vec{a}+\vec{b}\|^2 = \langle \vec{a}+\vec{b}, \vec{a}+\vec{b}\rangle$$

$$= \langle \vec{a},\vec{a}\rangle + \langle \vec{a},\vec{b}\rangle + \langle \vec{b},\vec{a}\rangle + \langle \vec{b},\vec{b}\rangle$$

$$= \|\vec{a}\|^2 + \langle \vec{a},\vec{b}\rangle + \overline{\langle \vec{a},\vec{b}\rangle} + \|\vec{b}\|^2$$

$$= \|\vec{a}\|^2 + 2\,\mathrm{Re}\langle \vec{a},\vec{b}\rangle + \|\vec{b}\|^2$$

$\langle\ ,\ \rangle$ Hermitian

Cauchy-Schwarz inequality :

$$\boxed{|\langle \vec{a},\vec{b}\rangle| \le \|a\|\,\|b\|} \quad (*)$$

Proof : assume $\vec{b} \neq \vec{0}$ ($\vec{a} \neq \vec{0}$) (if $\vec{b} = \vec{0}$ nothing to prove)

$$\langle \vec{a},\vec{b}\rangle = |\langle \vec{a},\vec{b}\rangle| e^{i\theta}$$

let $\vec{c} = e^{i\theta}\vec{b}$

$$\langle \vec{a},\vec{c}\rangle = \langle \vec{a}, e^{i\theta}\vec{b}\rangle = e^{-i\theta}\langle \vec{a},\vec{b}\rangle = |\langle \vec{a},\vec{b}\rangle|$$

$\in \mathbb{R}$ and $|\langle \vec{a},\vec{b}\rangle| = \langle \vec{a},\vec{c}\rangle = |\langle \vec{a},\vec{c}\,|\rangle$

moreover $\|\vec{c}\| = |e^{i\theta}|\,\|\vec{b}\| = \|\vec{b}\|$ hence to show (*) enough to show

$$\langle \vec{a}, \vec{c} \rangle \leq \|\vec{a}\|\,\|\vec{c}\|.$$

let $t \in \mathbb{R}$ and consider

$$\|\vec{a} + t\vec{c}\|^2 = \|\vec{a}\|^2 + 2\,\mathrm{Re}\langle \vec{a}, t\vec{c}\rangle + t^2\|\vec{c}\|^2 = \|\vec{a}\| + 2t\,\mathrm{Re}\langle \vec{a}, \vec{c}\rangle + t^2\|\vec{c}\|^2$$

$$= \underbrace{\|\vec{a}\|^2 + 2t\langle \vec{a}, \vec{c}\rangle + t^2\|\vec{c}\|^2}_{\geq 0}$$

quadratic polynomial of t

↑ cannot have 2 distinct roots, at most one.

discriminant $(2\langle \vec{a}, \vec{c}\rangle)^2 - 4\|\vec{c}\|^2\|\vec{a}\|^2 \leq 0$

$$4\langle \vec{a}, \vec{c}\rangle^2 \leq 4\|\vec{a}\|^2\|\vec{c}\|^2 \Rightarrow |\langle \vec{a}, \vec{c}\rangle| \leq \|\vec{a}\|\,\|\vec{c}\|$$

if $4\langle \vec{a}, \vec{c}\rangle = 4\|\vec{a}\|^2\|\vec{c}\|^2 \Rightarrow$ there is a unique root t_0 that is

$$\vec{a} = -t_0\vec{c}$$

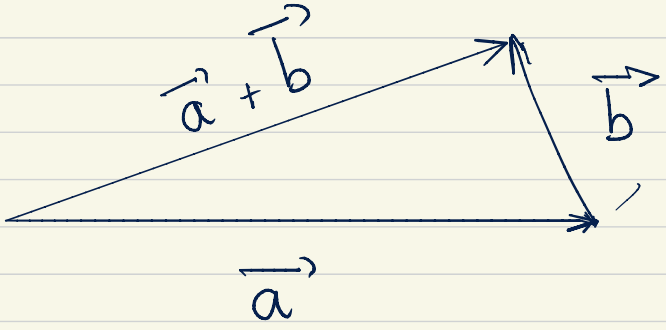
Triangle inequality : $\|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$

Proof : $\|\vec{a}+\vec{b}\|^2 = \|\vec{a}\|^2 + 2\mathrm{Re}\langle\vec{a},\vec{b}\rangle + \|\vec{b}\|^2$

Recall $\mathrm{Re}\langle\vec{a},\vec{b}\rangle \leq |\langle\vec{a},\vec{b}\rangle|$

$$\begin{aligned}\|\vec{a}+\vec{b}\|^2 &= \|\vec{a}\|^2 + 2|\langle\vec{a},\vec{b}\rangle| + \|\vec{b}\|^2 \\ &\leq \|\vec{a}\|^2 + 2\|\vec{a}\|\|\vec{b}\| + \|\vec{b}\|^2 \\ &\leq (\|\vec{a}\| + \|\vec{b}\|)^2\end{aligned}$$

Cauchy-Schwarz



Geometrically: one side of the triangle is no longer than the sum of the other two sides

Definition : We say that $\vec{a}$ & $\vec{b}$ in $\mathbb{C}^k$ are orthogonal if $\langle \vec{a}, \vec{b} \rangle = 0$

The vectors $\vec{a}_1, \vec{a}_2, \dots \vec{a}_n$ are mutually orthogonal if $\langle \vec{a}_i, \vec{a}_j \rangle = 0$ for all $i \neq j$.

The Pythagorean Theorem : If $\vec{a}_1, \vec{a}_2, \dots \vec{a}_n$ mutually orthogonal

$$\| \vec{a}_1 + \vec{a}_2 + \dots + \vec{a}_n \|^2 = \| \vec{a}_1 \|^2 + \dots + \| \vec{a}_n \|^2 \qquad (*)$$

Proof : By induction $n = 2$

$$\| \vec{a}_1 + \vec{a}_2 \|^2 = \| \vec{a}_1 \|^2 + 2\mathrm{Re} \underbrace{\langle \vec{a}_1, \vec{a}_2 \rangle}_{0} + \| \vec{a}_2 \|^2$$

assume $(*)$ holds for n, $\vec{a}_1 \dots \vec{a}_{n+1}$ mutually orthogonal

$$\| (\vec{a}_1 + \dots + \vec{a}_n) + \vec{a}_{n+1} \|^2 = \| \vec{a}_1 + \dots + \vec{a}_n \|^2 + 2\mathrm{Re} \underbrace{\langle \vec{a}_1 + \dots + \vec{a}_n, \vec{a}_{n+1} \rangle}_{\langle \vec{a}_1, \vec{a}_{n+1} \rangle + \dots \langle \vec{a}_n, \vec{a}_{n+1} \rangle} + \| \vec{a}_{n+1} \|^2$$

$\langle \vec{a}_1, \vec{a}_{n+1} \rangle = 0$, $\langle \vec{a}_n, \vec{a}_{n+1} \rangle = 0$

$$= \underbrace{\| \vec{a}_1 + \dots + \vec{a}_n \|^2}_{\text{by induction}} + \| \vec{a}_{n+1} \|^2$$

$$\text{by induction} = \| \vec{a}_1 \|^2 + \dots + \| \vec{a}_n \|^2 + \| \vec{a}_{n+1} \|^2$$

Note that the proofs of Cauchy - Schwarz, the Triangle Inequality and the Pythagorean theorem only depend on the fact that $\mathbb{C}^k$ is a linear (vector) space and that the inner product satisfies

i) $\langle \vec{a} + \vec{b}, \vec{c} \rangle = \langle \vec{a}, \vec{c} \rangle + \langle \vec{b}, \vec{c} \rangle$

iii) $\langle \alpha \vec{a}, \vec{b} \rangle = \alpha \langle \vec{a}, \vec{b} \rangle$

v) $\langle \vec{b}, \vec{a} \rangle = \overline{\langle \vec{a}, \vec{b} \rangle}$

and the definition of norm in terms of inner product

$$\|\vec{a}\| = \sqrt{\langle \vec{a}, \vec{a} \rangle}$$

Cauchy - Schwarz $\quad |\langle \vec{a}, \vec{b} \rangle| \leq \|\vec{a}\| \, \|\vec{b}\|$

Triangle inequality $\quad \|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$

Terminology: ① $\vec{u}$ is a unit vector if $\|\vec{u}\| = 1$

② a collection $\{\vec{a_1}, \vec{a_2}, \ldots\}$ of vectors is an orthogonal set if its elements are mutually orthogonal and nonzero; and an orthonormal set if its elements are mutually orthogonal unit vectors.

Note that any orthogonal set can be made into an orthonormal set by replacing each $\vec{a_j}$ by $\vec{a_j}/\|\vec{a_j}\|$

③ The Kronecker δ- symbol is

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

A set $\{\vec{a_1}, \vec{a_2} \ldots\}$ is orthonormal if and only if

$$\langle \vec{a_i}, \vec{a_j} \rangle = \delta_{ij}$$

The vectors in any orthogonal set $\{\vec{a_1}, \dots \vec{a_n}\}$ are linearly independent; that is if

$$c_1\vec{a_1} + \dots + c_n\vec{a_n} = 0 \quad \text{then} \quad c_j = 0 \text{ for each } 1 \le j \le n.$$

In fact since $\vec{a_j} \neq 0$ and

$$0 = \langle c_1\vec{a_1} + \dots + c_n\vec{a_n}, \vec{a_j} \rangle = c_1\langle \vec{a_1}, \vec{a_j}\rangle + \dots + c_n\langle \vec{a_n}, \vec{a_j}\rangle$$

$$0 = c_j \|\vec{a_j}\|^2 \implies c_j = 0$$

The number of vectors in any orthogonal set of $\mathbb{C}^k$ is at most k.

Example: $\vec{e_j} = (0, \dots 1, \dots 0)$ (1 in the jth position) $\quad \{\vec{e_j}\}_{j=1}^{k}$ orthonormal set

$$\vec{a} = (a_1, \dots, a_k) = a_1\vec{e_1} + \dots a_k\vec{e_k}$$

Proposition: Given an orthonormal set $\{\vec{u_1}, \dots, \vec{u_k}\}$ in $\mathbb{C}^k$ any vector can be written in an unique way as a linear combination of the $\vec{u_k}$'s.

Proof: Note that $\vec{a} - \langle \vec{a}, \vec{u}_1 \rangle \vec{u}_1 - \cdots - \langle \vec{a}, \vec{u}_k \rangle \vec{u}_k = \vec{v}$ is orthogonal to $\vec{u}_1, \vec{u}_2, \ldots \vec{u}_k$;

$$\langle \vec{v}, \vec{u}_\ell \rangle = \langle \vec{a}, \vec{u}_\ell \rangle - \langle \vec{a}, \vec{u}_1 \rangle \delta_{1\ell} \cdots - \langle \vec{a}, \vec{u}_k \rangle \delta_{k\ell}$$
$$= \langle \vec{a}, \vec{u}_\ell \rangle - \langle \vec{a}, \vec{u}_\ell \rangle = 0$$

If $\vec{v} \neq \vec{0}$ then $\{ \vec{u}_1, \cdots \vec{u}_k, \vec{v} \}$ is an orthogonal set in $\mathbb{C}^k$ with $(k+1)$ elements (this is a contradiction). Thus

$$\vec{a} = \langle \vec{a}, \vec{u}_1 \rangle \vec{u}_1 + \cdots + \langle \vec{a}, \vec{u}_k \rangle \vec{u}_k \quad \text{if}$$
$$\vec{a} = c_1 \vec{u}_1 + \cdots + c_k \vec{u}_k$$

then $(\langle \vec{a}, \vec{u}_1 \rangle - c_1) \vec{u}_1 + \cdots + (\langle \vec{a}, \vec{u}_k \rangle - c_k) \vec{u}_k = 0$

since $\{ \vec{u}_1, \ldots \vec{u}_k \}$ linearly independent then

$c_\ell = \langle \vec{a}, \vec{u}_\ell \rangle$ thus there is a unique representation

Note that by the Pythagorean theorem

$$\| \vec{a} \|^2 = |\langle \vec{a}, \vec{u}_1 \rangle|^2 \| \vec{u}_1 \|^2 + \cdots + |\langle \vec{a}, \vec{u}_k \rangle|^2 \| \vec{u}_k \|^2$$

Theorem: Let $\{\vec{u}_1, \ldots \vec{u}_k\}$ be an orthonormal set of k vectors in $\mathbb{C}^k$. For any $\vec{a} \in \mathbb{C}^k$

$$\vec{a} = \langle \vec{a}, \vec{u}_1 \rangle \vec{u}_1 + \cdots + \langle \vec{a}, \vec{u}_k \rangle \vec{u}_k$$

$$\|\vec{a}\|^2 = |\langle \vec{a}, \vec{u}_1 \rangle|^2 + \cdots + |\langle \vec{a}, \vec{u}_k \rangle|^2$$