(12) Suppose f is piecewise continuous on $[0, \pi]$ such that

$$f(\theta) = f(\pi - \theta)$$

a_n, b_n Fourier cosine and sine coefficients of f.

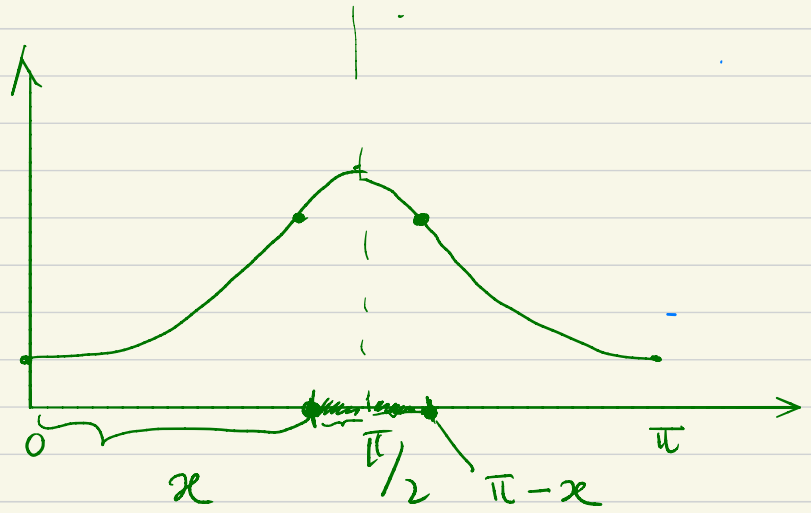
Show $a_{2n+1} = 0$ for all n

$b_{2n} = 0$ for all n

Note that $f(\theta) = f(\pi - \theta)$ means that f is symmetric with respect to the line $\theta = \frac{\pi}{2}$.

$f(0) = f(\pi - 0) = f(\pi)$

$f(x) = f(\pi - x)$



(12) Suppose f is piecewise continuous on $[0,\pi]$ such that

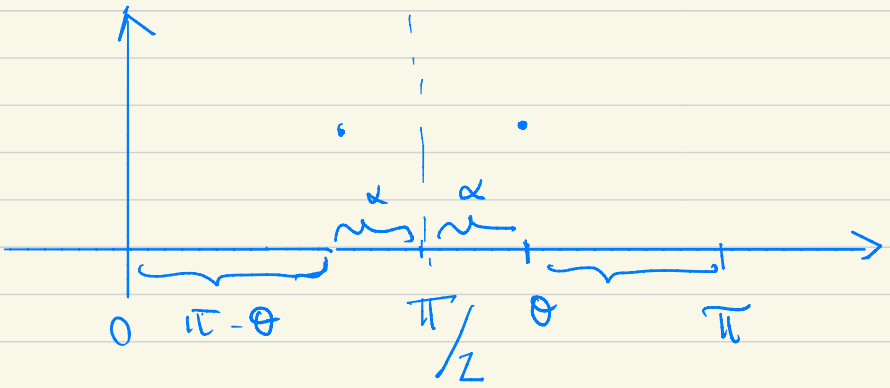
$$f(\theta) = f(\pi - \theta)$$

a_n, b_n Fourier cosine and sine coefficients of f.

Show $a_{2n+1} = 0$ for all n

$b_{2n} = 0$ for all n

Note that $f(\theta) = f(\pi - \theta)$ means that f is symmetric with respect to the line $\theta = \frac{\pi}{2}$. How do we check this?



If $\theta \in [\frac{\pi}{2}, \pi]$ $\theta = \frac{\pi}{2} + \alpha$ $\alpha > 0$

$\pi - \theta = \frac{\pi}{2} - \alpha$

$$f(\pi/2 + \alpha) = f(\pi - (\tfrac{\pi}{2} + \alpha))$$

$$\boxed{f(\tfrac{\pi}{2} + \alpha) = f(\tfrac{\pi}{2} - \alpha)}$$

Recall $a_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos n\theta\, d\theta$ & $b_n = \frac{2}{n}\int_0^{\pi} f(\theta)\sin n\theta\, d\theta$

$$a_n = \frac{2}{\pi}\int_0^{\pi/2} f(\theta)\cos n\theta\, d\theta + \frac{2}{\pi}\int_{\pi/2}^{\pi} f(\theta)\cos n\theta\, d\theta$$

change of variable $\theta = \pi/2 + \alpha$

$$= \frac{2}{\pi}\int_0^{\pi/2} f(\theta)\cos n\theta\, d\theta + \frac{2}{\pi}\int_0^{\pi/2} f\left(\frac{\pi}{2}+\alpha\right)\cos n\left(\frac{\pi}{2}+\alpha\right) d\alpha$$

change of variable $\tilde{\theta} = \frac{\pi}{2} - \alpha$

$$= \frac{2}{\pi}\int_0^{\pi/2} f(\theta)\cos n\theta\, d\theta + \frac{2}{\pi}\int_0^{\pi/2} f\left(\frac{\pi}{2}-\alpha\right)\cos n\left(\frac{\pi}{2}+\alpha\right) d\alpha$$

$$= \frac{2}{\pi}\int_0^{\pi/2} f(\theta)\cos n\theta\, d\theta + \frac{2}{\pi}\int_0^{\pi/2} f(\tilde{\theta})\cos n\left(\frac{\pi}{2}+\frac{\pi}{2}-\tilde{\theta}\right) d\tilde{\theta}$$

$\cos n(\pi - \tilde{\theta})$

$\cos n\pi \cos\tilde{\theta}n$ " $+ \sin n\pi \sin n\tilde{\theta}$

$\sin n\pi = 0$

$$a_n = \frac{2}{\pi}\int_0^{\pi/2} f(\theta)\cos n\theta\, d\theta + \frac{2}{\pi}\int_0^{\pi/2} f(\theta)\cos n\theta \underbrace{\cos n\pi}_{(-1)^n}\, d\theta$$

$$a_n = \underbrace{\frac{2}{\pi}\left(1+(-1)^n\right)}\int_0^{\pi/2} f(\theta)\cos n\theta\, d\theta \Rightarrow \boxed{a_n = 0 \text{ if } n \text{ odd}}$$

same manipulations as before

$$b_n = \frac{2}{\pi}\int_0^{\pi/2} f(\theta)\sin n\theta\, d\theta + \frac{2}{\pi}\int_0^{\pi} f(\theta)\underbrace{\sin n(\pi-\theta)}\, d\theta$$

$$\sin n\pi\cos n\theta - \cos n\pi \sin n\theta$$

$$0 \qquad - (-1)^n \sin n\theta$$

$$b_n = \underbrace{\frac{2}{\pi}\left(1-(-1)^n\right)}\int_0^{\pi/2} f(\theta)\sin n\theta\, d\theta$$

$$\Rightarrow \boxed{b_n = 0 \text{ if } n \text{ even}}$$

(7) Expand the function $f(x) = 1$ in a series of the form

$\sum b_n \sin \frac{n\pi x}{l}$ on $[0, 6\pi]$, here $l = 6\pi$.

- Additional problem **R**: Two rooms have been held at 20°C for so long that the thick wall (6π cm.) between them is also at 20°C throughout. Suddenly the air in both rooms is replaced by air at 0°C, and is held at that temperature. Assume that the interface between air and wall is perfectly conducting. (This is bad physics; we will replace this with a more realistic assumption in a couple of weeks.) Find the temperature in the wall as a function of position and time. Your solution will be in terms of the *thermal diffusivity* k. Hint: Problem # 7 in §2.4 page 48 should be useful.

(7) Expand the function $f(x) = 1$ in a series of the form

$\sum b_n \sin \frac{n\pi x}{l}$ on $[0, 6\pi]$, here $\boxed{l = 6\pi}$ define $\qquad \frac{n\pi x}{l} = \frac{nx}{6}$

let $x = \frac{l\theta}{\pi} = \frac{6\pi\theta}{\pi} = 6\theta$, $\quad g(\theta) = f(x) = f(6\theta)$

g is defined on $[0, \pi]$ the sine expansion corresponds to the sine expansion of f after the change of variables.

$$b_n = \frac{2}{\pi}\int_0^{\pi} g(\theta) \sin n\theta \, d\theta \underset{\theta = \frac{x}{6}}{=} \frac{2}{\pi}\int_0^{6\pi} f(x) \sin\left(\frac{nx}{6}\right) \frac{dx}{6}$$

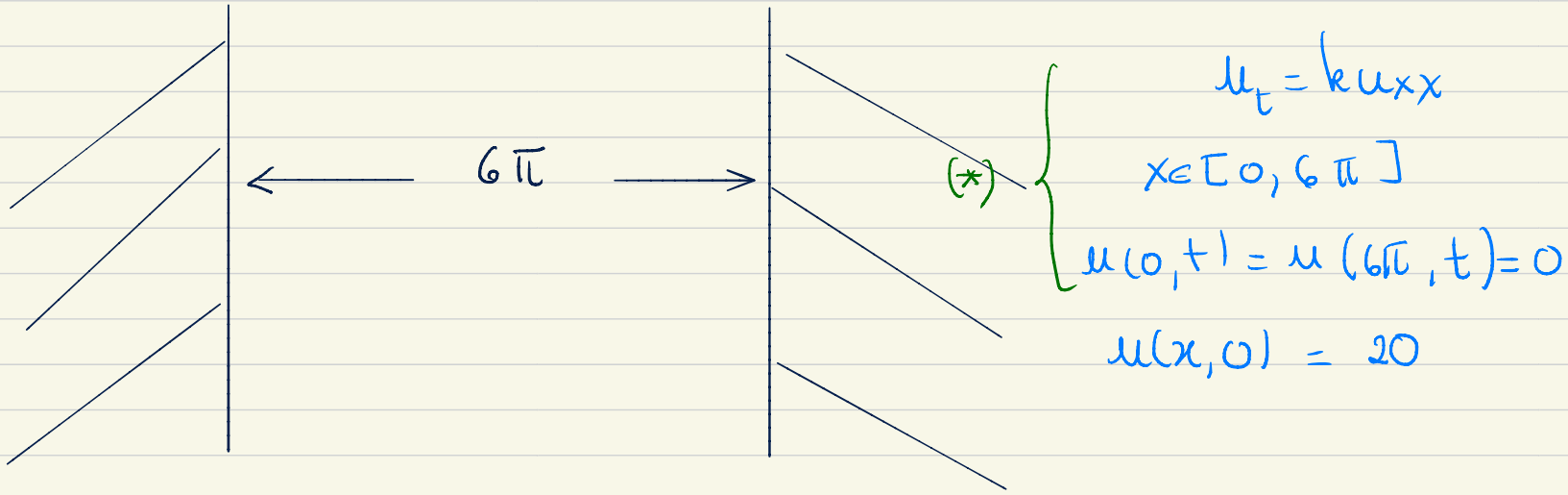
$$b_n = \frac{1}{3\pi}\int_0^{6\pi} \sin \frac{nx}{6} \, dx = \frac{1}{3\pi} \frac{6}{n} \left(-\cos \frac{nx}{6}\right)\Big|_0^{6\pi}$$

$$b_n = \frac{2}{n\pi}\left(-(-1)^n + 1\right) = \begin{cases} 0 & \text{if } n \text{ even} \\ 4/n\pi & \text{if } n \text{ odd} \end{cases}$$

$$b_n = \begin{cases} 0 & \text{if } n \text{ even} \\ 4/n\pi & \text{if } n \text{ odd} \end{cases} \Rightarrow \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin \frac{(2n+1)x}{6}$$

is the sine series for 1 in $[0, 6\pi]$

- Additional problem **R**: Two rooms have been held at 20°C for so long that the thick wall (6π cm.) between them is also at 20°C throughout. Suddenly the air in both rooms is replaced by air at 0°C, and is held at that temperature. Assume that the interface between air and wall is perfectly conducting. (This is bad physics; we will replace this with a more realistic assumption in a couple of weeks.) Find the temperature in the wall as a function of position and time. Your solution will be in terms of the *thermal diffusivity* k. Hint: Problem # 7 in §2.4 page 48 should be useful.



Solve

$$u_t = k u_{xx} \qquad \begin{cases} u(0,t) = u(6\pi, t) = 0 \\ u(x,0) = f(x) = 20 \end{cases} \qquad l = 6\pi$$

Recall that the solution is of the form

$$u = \sum_{n=1}^{\infty} a_n \exp\left(\frac{-n^2 \pi^2 k t}{(6\pi)^2} \right) \sin \frac{n\pi x}{6\pi}$$

$$u = \sum_{n=1}^{\infty} a_n \exp\left(-\frac{n^2 k t}{36} \right) \sin \frac{nx}{6}$$

$$20 = \sum_{n=1}^{\infty} a_n \sin \frac{nx}{6} = 20 \cdot 1 = 20 \cdot \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin \frac{(2n+1)x}{6}$$

↑ pb 7

$$a_n = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{80}{\pi} \frac{1}{2k+1} & \text{if } 2k+1 = n \end{cases}$$

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} \frac{80}{\pi} \frac{1}{2n+1} \exp\left(-\frac{(2n+1)^2 k t}{36} \right) \sin \frac{(2n+1)x}{6}}$$

The wave equation

Vibrating string of length l, fixed at both endpoints.

$$(*) \qquad u_{tt} = c^2 u_{xx} \qquad u(0,t) = u(l,t) = 0$$

Separation of variables: $u(x,t) = X(x)T(t)$ then $(*)$ becomes

$$X(x)T''(t) = c^2 X''(x)T(t) \qquad X(0)T(t) = X(l)T(t) = 0$$

$$\boxed{X(0) = X(l) = 0}$$

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = A \leftarrow \text{constant}$$

$$T''(t) = Ac^2 T(t) \qquad \text{and} \qquad X''(x) = AX(x)$$

If $A = 0$ then $T(t) = at + b$

$u \equiv 0$

$X(x) = cx + d$

$X(0) = d = 0$

$X(l) = cl = 0 \Rightarrow c = 0$

$A > 0$

$$T''(t) = Ac^2 T(t) \quad \& \quad X''(x) = AX(x)$$

$$T(t) = C_1 e^{\sqrt{A}ct} + C_2 e^{-\sqrt{A}ct} \quad \& \quad X(x) = \tilde{C}_1 e^{\sqrt{A}x} + \tilde{C}_2 e^{-\sqrt{A}x}$$

$$X(0) = \tilde{C}_1 + \tilde{C}_2 = 0 \quad \& \quad X(l) = \tilde{C}_1 e^{+\sqrt{A}l} + \tilde{C}_2 e^{-\sqrt{A}l} = 0$$

$$\Rightarrow \tilde{C}_1 \underbrace{\left(e^{\sqrt{A}l} - e^{-\sqrt{A}l}\right)}_{\neq 0} = 0 \Rightarrow \tilde{C}_1 = 0 \Rightarrow \tilde{C}_2 = 0$$

replacing $\tilde{C}_2$ by $-\tilde{C}_1$

$u \equiv 0$.

$A < 0$ let $A = -\lambda^2$ then

$$T''(t) = -\lambda^2 c^2 T(t) \quad \& \quad X''(x) = -\lambda^2 X(x)$$

$$T(t) = C_1 \cos(\lambda c t) + C_2 \sin(\lambda c t) \quad \& \quad X(x) = \tilde{C}_1 \cos \lambda x + \tilde{C}_2 \sin \lambda x$$

$$X(0) = \tilde{C}_1 = 0$$

$$X(l) = \tilde{C}_2 \sin \lambda l = 0$$

then $\lambda l = n\pi \Rightarrow \boxed{\lambda = \frac{n\pi}{l}}$ $\quad n = 0, 1, 2 \ldots$

thus $u_n(x,t) = \left(a_n \cos \frac{n\pi c t}{l} + b_n \sin \frac{n\pi c t}{l}\right) \sin \frac{n\pi x}{l}$

We obtain the formal solution

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi c t}{l} + b_n \sin \frac{n\pi c t}{l} \right) \sin \frac{n\pi x}{l}$$

Recall the appropriate type of initial conditions

$$u(x,0) = f(x) \quad \& \quad \frac{\partial u}{\partial t}(x,0) = g(x)$$

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \quad \& \quad g(x) = \sum_{n=1}^{\infty} \underbrace{\frac{n\pi c}{l} b_n}_{B_n} \sin \frac{n\pi x}{l}$$

$t=0$

because (formal) differentiation yields

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} \left(a_n \frac{n\pi c}{l} \underbrace{\cancel{\left(-\sin \frac{n\pi c t}{l}\right)}}_{0} + b_n \frac{n\pi c}{l} \underbrace{\cos\left(\frac{n\pi c t}{l}\right)}_{1} \right) \sin \frac{n\pi x}{l}$$

$t=0$

Summarizing: the formal solution to

(*) $u_{tt} = c^2 u_{xx}$, $u(x,0) = f(x)$, $u_t(x,0) = g(x)$, $u(0,t) = u(l,t) = 0$

with

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \& \quad g(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}$$

is

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \left(b_n \cos \frac{n\pi ct}{l} + \frac{l B_n}{n\pi c} \sin \frac{n\pi ct}{l} \right)}$$

Question: is $u(x,t)$ a solution to (*)? What happens when we differentiate twice? w.r.t x

$$\left(\frac{\sin n\pi x}{l} \right)'' = \left(\frac{n\pi}{l} \cos \frac{n\pi x}{l} \right)' = -\frac{n^2\pi^2}{l^2} \sin \frac{n\pi x}{l}$$

$$b_n \rightsquigarrow -\frac{n^2\pi^2}{l^2} b_n$$

We can avoid this by assuming that f and g are smooth enough.
For example if:

(1) f is of class $C^{(3)}$, $f^{(3)}$ piecewise smooth; $f(0) = f(l) = f''(0) = f''(l) = 0$

(2) g is of class $C^{(2)}$, g'' piecewise smooth; $g(0) = g(l) = g''(0) = g''(l) = 0$

then the odd extensions of f and g will be $C^{(3)}$ and $C^{(2)}$ resp.
(check this) and then we know that

$$|b_n| \leq C n^{-4} \quad \text{and} \quad |B_n| \leq C n^{-3}$$

Hence $$\left| b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \right| \leq |b_n| \leq C n^{-4}$$

and $$\left| \frac{B_n l}{n\pi c} \sin \frac{n\pi x}{l} \sin \frac{n\pi ct}{l} \right| \leq C \frac{|B_n|}{n} \leq C' n^{-4}$$

What happens when we differentiate twice? all coefficients are bounded $\leq C n^{-2}$

It is physically reasonable to assume f and g continuous maybe piecewise smooth but additional regularity is not meaningful.

Recall

$$\sin a \cos b = \frac{1}{2}\left[\sin(a+b) + \sin(a-b)\right]$$

$$\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$

assume f & g piecewise smooth with the following expansions in $[0, l]$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \& \quad g(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \left(b_n \cos \frac{n\pi ct}{l} + \frac{l B_n}{n\pi c} \sin \frac{n\pi ct}{l} \right)$$

$$\frac{1}{2} \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l}(x+ct) + \frac{1}{2} \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l}(x-ct)$$

$$\sum_{n=1}^{\infty} \frac{B_n}{n\pi} \cos \frac{n\pi}{l}(x-ct) - \frac{1}{2c} \sum_{n=1}^{\infty} \frac{B_n}{n\pi} \cos \frac{n\pi}{l}(x+ct)$$

The odd extension of f and g (recall $f(0)=f(l)=0$ & $g(0)=g(l)=0$) is $2l$-periodic then

$$u(x,t) = \frac{1}{2}\underbrace{\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}(x+ct)}_{f(x+ct)} + \frac{1}{2}\underbrace{\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}(x-ct)}_{f(x-ct)}$$

$$+ \frac{1}{2c}\underbrace{\sum_{n=1}^{\infty} \frac{B_n}{n\pi}\cos\frac{n\pi}{l}(x-ct)}_{G(x-ct)} - \frac{1}{2c}\underbrace{\sum_{n=1}^{\infty} \frac{B_n}{n\pi}\cos\frac{n\pi}{l}(x+ct)}_{G(x+ct)}$$

where $G' = g$.

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\left[G(x-ct) - G(x+ct)\right]$$

If f is C^2 and g is C^1 then G is C^2.

$$\frac{\partial^2 f}{\partial x^2} f(x\pm ct) = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} f(x\pm ct)$$ and same for G.

Physical difference between solutions to heat equation and wave equation

$$u(x,t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \exp\left(\frac{-n^2\pi^2 kt}{l^2}\right) \cos\frac{n\pi x}{l}$$

Diffusion processes tend to smooth out singularities in the initial data; the solution becomes smoother as t increases

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left(b_n \cos\frac{n\pi ct}{l} + \frac{lB_n}{n\pi c}\sin\frac{n\pi ct}{l}\right)$$

Wave motion propagates singularities; any corners in the initial configuration of a vibrating string will not disappear they move back and forth - No improvement in the rate of convergence of the solution