Theorem: Suppose f is 2π-periodic, and piecewise smooth. Let a_n, b_n and c_n be the Fourier coefficients of f, and let a_n', b_n' and c_n' be the Fourier coefficients of f'. Then

$$\boxed{a_n' = n b_n \quad , \quad b_n' = -n a_n \quad \text{and} \quad c_n' = in c_n .}$$

Theorem: Suppose f is 2π-periodic and piecewise continuous with Fourier coefficients a_n, b_n and c_n. Let

$$F(\theta) = \int_0^\theta f(t)\,dt \qquad \text{and} \qquad c_0 = \frac{1}{2}a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

then

$$\sum_{n\neq 0} \frac{c_n}{in} e^{in\theta}$$ is the Fourier series of $G(\theta) = F(\theta) - c_0\theta$

Theorem: If f is 2π-periodic, continuous, and piecewise smooth then the Fourier series of f converges uniformly and absolutely on $\mathbb{R}$.

Theorem: Suppose f is 2π-periodic. If f is of class $C^{(k-1)}$ and $f^{(k-1)}$ is piecewise smooth (i.e. $f^{(k)}$ exists except at finitely many points in each bounded interval and it is piecewise continuous) then the Fourier coefficients of f (a_n, b_n, c_n) satisfy

$$\sum_{n=1}^{\infty} |n^k a_n|^2 < \infty, \quad \sum_{n=1}^{\infty} |n^k b_n|^2 < \infty \quad \& \quad \sum_{n=-\infty}^{\infty} |n^k c_n|^2 < \infty$$

In particular

$$n^k a_n \to 0 \quad n^k b_n \to 0 \text{ and } n^k c_n \to 0 \quad |n| \to \infty$$

On the other hand, suppose the Fourier coefficients c_n $n \neq 0$ satisfy $|c_n| \leq C|n|^{-(k+\alpha)}$ (eq $|a_n| \leq Cn^{-(k+\alpha)}$ $|b_n| \leq C n^{-(k+\alpha)}$) for some $C > 0$ and $\alpha > 1$, then $f \in C^{(k)}$.

$f \in C^{(k-1)}$, $f, f', f'', \ldots, f^{(k-1)}$ continuous.

Proof: For the 1st part we apply the previous result k times, then if $c_n^{(j)}$ the Fourier coefficient of $f^{(j)}$, then

$$c_n^{(k)} = in\, c_n^{(k-1)} = (in)^k c_n, \text{ by Bessel's inequality}$$

$$\sum |c_n^{(k)}|^2 = \sum |n|^{2k} |c_n|^2 < \infty \quad \text{(similarly for an \& bn)}$$

For the 2nd part

$$\sum_{\substack{n\neq 0\\ -\infty}}^{\infty} |n|^j |c_n| \le C \sum_{\substack{n\neq 0\\ -\infty}}^{\infty} |n|^j |n|^{-k-\alpha} \le 2C \sum_{n\neq 0}^{\infty} |n|^{-\alpha}$$

for $j \le k$ as $|n|^{j-k} \le 1$

Thus by the Weierstrass M-test the series

$$\sum_{n=-\infty}^{\infty} (in)^j c_n e^{in\theta} \text{ is absolutely and uniformly convergent for } j \le k.$$

The uniform limit of continuous functions is continuous thus $f^{(j)}$ is continuous for all $j \le k$.

§ 2.2 page 37 # 1 : All functions defined in $[-\pi, \pi]$

$f(\theta)$	continuous	piecewise continuous	piecewise smooth
$\frac{1}{\sin\theta}$			
$(\sin\theta)^{1/3}$			
$(\sin\theta)^{4/3}$			
$\cos\theta \quad \theta > 0$ $-\cos\theta \quad \theta < 0$			
$\sin\theta \quad \theta > 0$ $\sin 2\theta \quad \theta < 0$			
$(\sin\theta)^{1/5} \quad \theta < \frac{\pi}{2}$ $\cos\theta \quad \theta \geq \pi/2$			

§ 2.2 page 37 # 1 : All functions defined in $[-\pi, \pi]$

$f(\theta)$	continuous	piecewise continuous	piecewise smooth
$\frac{1}{\sin\theta}$	no	$\lim_{\theta \to 0^+} (\sin\theta)^{-1} = +\infty$ no	no
$(\sin\theta)^{1/3}$	yes	yes	$\frac{1}{3}\cos\theta \sin\theta^{-2/3}$ no
$(\sin\theta)^{4/3}$	yes	yes	$\frac{4}{3}\cos\theta \sin\theta^{1/3}$ yes
$\cos\theta \quad \theta > 0$ $-\cos\theta \quad \theta < 0$	no	$\lim_{\theta \to 0^+} = 1$ $\lim_{\theta \to 0^-} = -1$ yes	yes
$\sin\theta \quad \theta > 0$ $\sin 2\theta \quad \theta < 0$	yes	yes	$\cos\theta \quad \theta > 0$ yes $2\cos 2\theta \quad \theta < 0$
$(\sin\theta)^{1/5} \quad \theta < \frac{\pi}{2}$ $\cos\theta \quad \theta \geq \pi/2$	no	$\lim_{\theta \to \pi/2^-} = 1$ $\lim_{\theta \to \pi/2^+} = 0$	$\frac{1}{5}\sin\theta^{-4/5}\cos\theta \quad \theta < \frac{\pi}{2}$ pb 0 $\sin\theta \quad \theta > \pi/2$ no

§ 2.2 page 37 #2

6 - $$f(\theta) = \begin{cases} -1 & -\pi < \theta < 0 \\ 1 & 0 < \theta < \pi \end{cases}$$

7 - $$f(\theta) = \begin{cases} 0 & -\pi < \theta < 0 \\ 1 & 0 < \theta < \pi \end{cases}$$

12 - $$f(\theta) = \begin{cases} (2a)^{-1} & |\theta| < a \\ 0 & a < |\theta| < \pi \end{cases}$$

18 - $f(\theta) = e^{b\theta}$ for $-\pi < \theta < \pi$

§ 2.2 page 37 #2

	Points of discontinuity	Limits
6. $f(\theta) = \begin{cases} -1 & -\pi < \theta < 0 \\ 1 & 0 < \theta < \pi \end{cases}$	$k\pi$	0
7. $f(\theta) = \begin{cases} 0 & -\pi < \theta < 0 \\ 1 & 0 < \theta < \pi \end{cases}$	$k\pi$	$\frac{1}{2}$
12. $f(\theta) = \begin{cases} (2a)^{-1} & \lvert\theta\rvert < a \\ 0 & a < \lvert\theta\rvert < \pi \end{cases}$	$-a + 2k\pi$ $a + 2k\pi$	$\frac{1}{4a}$
18. $f(\theta) = e^{b\theta}$ for $-\pi < \theta < \pi$	$(2k+1)\pi$	$\frac{e^{-b\pi} + e^{b\pi}}{2}$

(3)

Using the fact that the Fourier series of $f(\theta) = |\sin\theta|$ is

$$\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2n\theta}{4n^2-1}$$ compute

$$\sum_{n=1}^{\infty}\frac{1}{4n^2-1} \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}$$

$\theta = 0$ $\quad \cos 2n\theta = \cos 0 = 1$

$$\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = 0$$

$$\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{2}{\pi}\cdot\frac{\pi}{4} = \frac{1}{2}$$

$\theta = \pi/2$

$\cos 2n\theta = \cos n\pi = (-1)^n$

$$\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1} = 1$$

$$-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1} = 1 - \frac{2}{\pi}$$

$$-\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1} = \frac{\pi}{4}\left(\frac{\pi-2}{\pi}\right)$$

$-(-1)^n = (-1)^{n+1}$

Using the fact that the Fourier series of $f(\theta) = |\sin\theta|$ is

$$\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2n\theta}{4n^2-1}$$ compute

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2-1}$$

For $\theta = 0$ $\quad \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2n0}{4n^2-1} = 0 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$

For $\theta = \pi/2$ $\quad \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos n\pi}{4n^2-1} = 1$

$$-\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2-1} = \left(1 - \frac{2}{\pi}\right)\frac{\pi}{4} = \frac{\pi-2}{4}$$

Compute the Fourier cosine series & the Fourier sine series of the functions below given in the interval $[0,\pi]$. Use the table. To what values do these series converge when $\theta = 0$ and $\theta = \pi$?

(1A) $f(\theta) = \theta$

(1) $f(\theta) = 1$

(3) $f(\theta) = \sin\theta$

Recall cosine Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta$

Fourier coefficient of even extension $\rightsquigarrow$ $a_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos n\theta \, d\theta$

Sine Fourier series $\sum_{n=1}^{\infty} b_n \sin n\theta$

Fourier coefficient of odd extension $\rightsquigarrow$ $b_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin n\theta \, d\theta$

① $f(\theta) = 1$

Even extension

$f(\theta) = 1$ in $[-\pi, \pi]$

$a_n = 0$ for $n \geq 1$

$a_0 = 2$

limit at 0 and π is 1

odd extension

$$f(\theta) = \begin{cases} 1 & \text{in } [0, \pi) \\ -1 & \text{in } (-\pi, 0) \end{cases}$$

function ⑥

$$\frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\theta}{2n+1}$$

at 0 & π the limit is 0!

(1A) $f(\theta) = \theta$

Even extension

$$f_{\text{even}}(\theta) = \begin{cases} \theta & \text{in } [0, \pi] \\ -\theta & \text{in } [-\pi, 0) \end{cases}$$

$$f_{\text{even}}(\theta) = |\theta| \quad \text{in } [-\pi, \pi]$$

function (2) in the table

$$\frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)\theta}{(2n-1)^2}$$

limit at 0 is 0

limit at π is π

because f_{even} continuous

Odd extension

$$f_{\text{odd}}(\theta) = \begin{cases} \theta & \text{in } [0, \pi] \\ \theta & \text{in } [-\pi, 0) \end{cases}$$

$$f_{\text{odd}}(\theta) = \theta \quad \text{in } [-\pi, \pi]$$

function (1) in the table

$$2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\theta$$

limit at 0 is 0 because f_{odd} is continuous at 0

limit at π is $\dfrac{f_{\text{odd}}(\pi+) + f_{\text{odd}}(\pi-)}{2}$

$$= \frac{-\pi + \pi}{2} = 0$$

③ $f(\theta) = \sin\theta$

Even extension

$$f_{even}(\theta) = \begin{cases} \sin\theta & 0 \le \theta \le \pi \\ -\sin\theta & -\pi \le \theta < 0 \end{cases}$$

$$f_{even}(\theta) = |\sin\theta| \quad \text{in } [-\pi, \pi]$$

Function (8) in the table

$$\frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2n\theta}{4n^2 - 1}$$

limit at 0 is 0 & limit at π is 0 because f_{even} is continuous

Odd extension

$$f_{odd}(\theta) = \begin{cases} \sin\theta & 0 \le \theta \le \pi \\ \sin\theta & -\pi \le \theta < 0 \end{cases}$$

$f_{odd}(\theta) = \sin\theta$ in $[-\pi, \pi]$

↑ this is the sine Fourier series

limit at 0 is 0 and limit at π is 0 because f_{odd} is continuous.

Back to the heat equation: if $\begin{cases} u_t = k u_{xx} \\ u(x,0) = f(x) \end{cases}$ for $x \in [0, l]$

with $u(0,t) = u(l,t) = 0$

a candidate solution was

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp\left(\frac{-n^2\pi^2 k t}{l^2}\right) \sin\frac{n\pi x}{l} \quad \text{where}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Questions left open: ① Can f be expressed as a sine series?
② Does this series actually define a solution with the given boundary conditions?

① yes if f is piecewise smooth in $[0,l]$. In fact note that f can be extended to be $2l$ periodic, since $f(0) = f(l) = 0$ the odd extension is continuous at 0.

Consider $x = \frac{l\theta}{\pi}$, $g(\theta) = f(x) = f\left(\frac{l\theta}{\pi}\right)$ is 2π-periodic.

in fact

$$g(\theta + 2\pi) = f\left(\frac{l}{\pi}(\theta + 2\pi)\right) = f\left(\frac{l\theta}{\pi} + 2l\right) = f\left(\frac{l\theta}{\pi}\right) = g(\theta)$$

and g is piecewise smooth, and odd thus $a_n = 0$ and

$$b_n = \frac{2}{\pi}\int_0^{\pi} g(\theta)\sin n\theta \, d\theta = \frac{2}{\pi}\int_0^{\pi} f\left(\frac{l\theta}{\pi}\right)\sin n\theta \, d\theta \qquad x = \frac{l\theta}{\pi}$$

and its Fourier series converges $= \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}\,dx$

to $\sum_{n=1}^{\infty} b_n \sin n\theta = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$.

(2) $u_n(x,t) = \exp\left(\frac{-n^2\pi^2 kt}{l^2}\right)\sin\frac{n\pi x}{l}$ is a solution of the heat equation with the correct boundary data by construction.

For $t > 0$ since $b_n \to 0$ (by Bessel's inequality) for n large enough $|b_n| \leq 1$ $(n \geq N)$ then

$$\underbrace{u(x,t)} = \sum_{n=1}^{\infty} b_n \underbrace{\exp\left(\frac{-n^2\pi^2 kt}{l^2}\right)} \sin\frac{n\pi x}{l}$$

$b_n \to 0$

$n \to \infty$

very fast

$|b_n| \leq 1 \qquad n \geq N$

large

$$\left| b_n \exp\left(-\frac{n^2\pi^2 k t}{l^2}\right) \sin\frac{n\pi x}{l} \right| \le \exp\left(-\frac{n^2\pi^2 k t}{l^2}\right) \le \underbrace{e^{-n^2\delta}}$$

where $\delta = \frac{\pi^2 k \varepsilon}{l^2}$ for $t \ge \varepsilon$ for $n \ge N$

$$\frac{\partial}{\partial t} u_n(x,t) = -n^2\pi^2 k \, u_n(x,t) = k \frac{\partial^2 u_n(x,t)}{\partial x^2}$$

$$\left| b_n \frac{n^2\pi^2 k}{l^2} u_n(x,t) \right| \le \frac{n^2\pi^2 k}{l^2} \underbrace{e^{-n^2\delta}}$$

Since $\sum_{n=1}^{\infty} n^l e^{-n^2\delta}$ converges for any l, by the Weierstrass M-test the series converges absolutely and uniformly in $[0,l]$ and $t \ge \varepsilon$. Hence the differentiation of the series twice is permissible, u is a solution of the heat equation in $[0,l] \times (0,\infty)$. What happens as $t = 0$?

If f is <u>continuous</u> & piecewise smooth the odd extension is also

continuous and the Fourier series of f converges uniformly and absolutely, thus $\sum |b_n| < \infty$, the Weierstrass M-test shows that $u(x,t)$ converges uniformly and absolutely in $[0,l]$, $t \geq 0$

u continuous and $u(x,t) \longrightarrow u(x,0) = f(x)$

$t \to 0$

Uniqueness of the solution: via uniqueness of the Fourier transform.