The partial sum for the Fourier series of f is

$$S_N^f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{N}(a_n \cos n\theta + b_n \sin n\theta) = \sum_{n=-N}^{N} c_n e^{in\theta}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\psi)\cos n\psi \, d\psi \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\psi)\sin n\psi \, d\psi$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\psi)e^{-in\psi} d\psi$$

$$S_N^f(\theta) = \int_{-\pi}^{\pi} f(\theta+\phi)D_N(\phi)\,d\phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta+\phi)\,\frac{\sin\left(N+\frac{1}{2}\right)\phi}{\sin\frac{\phi}{2}}\,d\phi$$

$$D_N(\phi) = \frac{1}{2\pi}\sum_{n=-N}^{N} e^{in\phi} = \frac{1}{2\pi}\,\frac{\sin\left(N+\frac{1}{2}\right)\phi}{\sin \phi/2}$$ is the N-Dirichlet kernel

Recall that $\int_{-\pi}^{0} D_N(\phi)\,d\phi = \int_{0}^{\pi} D_N(\phi)\,d\phi = \frac{1}{2}\int_{-\pi}^{\pi} D_N(\phi)\,d\phi = \frac{1}{2}$

for all N.

Theorem: If f is 2π-periodic and piecewise smooth on $\mathbb{R}$ then

$$\lim_{N\to\infty} S_N^f(\theta) = \frac{1}{2}\left[f(\theta-) + f(\theta+)\right]$$

Proof:

$$S_N^f(\theta) - \frac{1}{2}\left[f(\theta-) + f(\theta+)\right]$$

$$= \int_{-\pi}^{0} f(\theta+\phi) D_N(\phi)\,d\phi - \frac{1}{2} f(\theta-)$$

$$+ \int_{0}^{\pi} f(\theta+\phi) D_N(\phi)\,d\phi - \frac{1}{2} f(\theta+)$$

$$\int_{-\pi}^{0} f(\theta+\phi)\, D_N(\phi)\, d\phi - \frac{1}{2} f(\theta-) = \int_{-\pi}^{0} f(\theta+\phi)\, D_N(\phi)\, d\phi - f(\theta-) \int_{-\pi}^{0} D_N(\phi)\, d\phi$$

$$= \int_{-\pi}^{0} \left[f(\theta+\Phi) - f(\theta-) \right] D_N(\phi)\, d\phi$$

$$\int_{0}^{\pi} f(\theta+\phi)\, D_N(\phi)\, d\phi - \frac{1}{2} f(\theta+) = \int_{0}^{\pi} f(\theta+\phi)\, D_N(\phi)\, d\phi - f(\theta+) \int_{0}^{\pi} D_N(\phi)\, d\phi$$

$$= \int_{0}^{\pi} \left[f(\theta+\Phi) - f(\theta+) \right] D_N(\phi)\, d\phi$$

$$S_N^f(\theta) - \frac{1}{2} \left[f(\theta-) + f(\theta+) \right] = \int_{-\pi}^{0} \left[f(\theta+\Phi) - f(\theta-) \right] D_N(\phi)\, d\phi$$

$$+ \int_{0}^{\pi} \left[f(\theta+\Phi) - f(\theta+) \right] D_N(\phi)\, d\phi$$

$$\int_0^{\pi} [f(\theta+\phi) - f(\theta+) D_N(\phi)\, d\phi = \frac{1}{2\pi} \int_0^{\pi} \frac{f(\theta+\phi) - f(\theta+)}{e^{i\phi} - 1} \cdot \left(e^{i(N+1)\phi} - e^{-iN\phi}\right) d\phi$$

$$\int_{-\pi}^{0} [f(\theta+\phi) - f(\theta-)] D_N(\phi)\, d\phi = \frac{1}{2\pi} \int_{-\pi}^{0} \frac{f(\theta+\phi) - f(\theta-)}{e^{i\phi} - 1} \left(e^{i(N+1)\phi} - e^{-iN\phi}\right) d\phi$$

let

$$g(\phi) = \begin{cases} \dfrac{f(\theta+\phi) - f(\theta-)}{e^{i\phi} - 1} & -\pi < \phi < 0 \\ \dfrac{f(\theta+\phi) - f(\theta+)}{e^{i\phi} - 1} & 0 < \phi < \pi \end{cases}$$

g is smooth where f is in $[-\pi, \pi]$ minus $\phi = 0$

$$\lim_{\phi \to 0^+} g(\phi) = \lim_{\phi \to 0^+} \frac{f(\theta+\phi) - f(\theta+)}{e^{i\phi} - 1} \overset{\text{l'Hôpital rule}}{=} \lim_{\phi \to 0^+} \frac{f'(\theta+\phi)}{ie^{i\phi}} = \frac{f'(\theta+)}{i}$$

$$g(\phi_\pm) = -i f'(\theta_\pm), \quad g \in PC[-\pi, \pi]$$

$$S_N^f(\theta) - \frac{1}{2}\left(f(\theta+) + f(\theta-)\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\left(e^{i(N+1)\phi} - e^{-iN\phi}\right)d\phi$$

$$= c_{-(N+1)} - c_N \quad \text{where}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\, e^{in\theta}\, d\theta \quad \text{the Fourier coefficient of } g$$

by Bessel's inequality $\lim_{N\to\infty} c_N = \lim_{N\to\infty} c_{-(N+1)} = 0$

thus

$$\lim_{N\to\infty} S_N^f(\theta) = \frac{1}{2}\left(f(\theta+) + f(\theta-)\right) \qquad \blacksquare$$

Remark : If f is continuous at θ,

$$\lim_{N\to\infty} S_N^f(\theta) = f(\theta).$$

$$\frac{\pi}{2} + \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos(2k+1)\theta}{(2k+1)^2} = \begin{cases} \pi - \theta & 0 \le \theta \le \pi \\ \pi + \theta & -\pi \le \theta \le 0 \end{cases}$$

Find $\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$ // $\theta = 0$

$$\frac{\pi}{2} + \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \pi$$

$$\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \pi/2 \quad \Rightarrow \quad \boxed{\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \pi^2/8}$$

$$\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin(2k+1)\theta}{2k+1} = \begin{cases} -1 & -\pi < \theta < 0 \\ 1 & 0 < \theta < \pi \\ 0 & \theta = 0 \end{cases}$$

Find $\pi/4 = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$ | $\theta = \pi/2$

$$\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin(2k+1)\pi/2}{2k+1}$$

$\sin(2k+1)\pi/2 = \sin(k\pi + \pi/2) = \cos k\pi \sin\pi/2$ — trig identity

$= (-1)^k$

$$\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} = 1$$

Derivatives, Integrals & Uniform Convergence

Questions: If $f \in PS(\mathbb{R})$ what is the relationship between the Fourier series for f, f' and $\int_0^x f(t)\,dt$?

Note that the fundamental theorem of calculus applies to piecewise smooth functions. Recall if f continuous in $[a,b]$ and differentiable in (a,b) then

$$\int_a^b f'(t)\,dt = f(b) - f(a)$$

Assume f is bounded in $[a,b]$; there is $c \in (a,b)$ such that f continuous and differentiable in $(a,c) \cup (c,b)$

$$\int_a^b f'(t)\,dt = \int_a^c f'(t)\,dt + \int_c^b f'(t)\,dt = (f(c) - f(a)) + (f(b) - f(c))$$
$$= f(b) - f(a)$$

Theorem: Suppose f is 2π-periodic, and piecewise smooth. Let a_n, b_n and c_n be the Fourier coefficients of f, and let a_n', b_n' and c_n' be the Fourier coefficients of f'. Then

$$\boxed{a_n' = n b_n \quad , \quad b_n' = -n a_n \quad \text{and} \quad c_n' = in c_n}$$

Proof:

$$c_n' = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(\theta) e^{-in\theta}\, d\theta = \frac{1}{2\pi} f(\theta) e^{-in\theta}\Big|_{-\pi}^{\pi}$$

↑ integration by parts

$$- \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)(-ine^{-in\theta})\, d\theta$$

$$= \frac{1}{2\pi}\left(f(\pi) e^{-in\pi} - f(-\pi) e^{+in\pi} \right) + in \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-in\theta}\, d\theta$$

since f 2π periodic $f(\pi) = f(-\pi)$ (so the first term is 0)

and $e^{-in\pi} = \cos(-n\pi) = \cos(n\pi) = e^{in\pi}$

$in\, c_n$

$$\boxed{c_n' = in c_n}$$

$$a_n' = c_n' + c_{-n}' = in(c_n - c_{-n}) = n b_n$$

$$b_n' = i(c_n' - c_{-n}') = -n(c_n + c_{-n}) = -n a_n$$

Theorem: Suppose f is 2π-periodic and piecewise continuous with Fourier coefficients a_n, b_n and c_n. Let

$$F(\theta) = \int_0^\theta f(t)\,dt \qquad \text{and} \qquad c_0 = \frac{1}{2}a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

then

$$\sum_{n\neq 0} \frac{c_n}{in} e^{in\theta}$$ is the Fourier series of $G(\theta) = F(\theta) - c_0\theta$

Proof: Note that in general F is not 2π-periodic, but G is

$$G(\theta+2\pi) - G(\theta) = F(\theta+2\pi) - c_0(\theta+2\pi) - (F(\theta) - c_0\theta)$$

$$= F(\theta+2\pi) - F(\theta) - 2\pi c_0 = \int_\theta^{\theta+2\pi} f(t)\,dt - 2\pi c_0$$

$$= \int_\theta^{\theta+2\pi} f(t)\,dt - \int_{-\pi}^{\pi} f(t)\,dt = 0$$ (by lemma for 2π-periodic functions.

G is 2π-periodic, piecewise smooth with coefficients α_n, β_n and γ_n. Note that the Fourier coefficients of $G' = f - c_0$ are $\gamma'_0 = 0$ and $\gamma'_n = c_n$ thus by the previous theorem

$c_n = \gamma'_n = in\gamma_n$; $\gamma_0 = 0$ and $c_n = in\gamma_n$ thus $\gamma_n = \dfrac{c_n}{in}$ $n \neq 0$

Recall: if $f \in PS(\mathbb{R})$ the Fourier series of f converges pointwise.

Definitions: ① We say that $\sum_{n=1}^{\infty} g_n$ converges to g (pointwise) in S if

$$\lim_{N \to \infty} \sum_{n=1}^{N} g_n(x) = g(x) \quad \text{for all } x \in S$$

② $\sum_{n=1}^{\infty} g_n$ converges absolutely if $\sum_{n=1}^{\infty} |g_n|$ converges

③ $\sum_{n=1}^{\infty} g_n$ converges uniformly in S to g if

$$\sup_{x \in S} \left| g(x) - \sum_{n=1}^{N} g_n(x) \right| \longrightarrow 0 \quad \text{as } N \to \infty$$

Weierstrass M-test: if there exists a sequence M_n of positive constants such that

$$|g_n(x)| \le M_n \quad \text{for } x \in S \quad \text{and} \quad \sum_{n=1}^{\infty} M_n < \infty$$

then the series $\sum_{n=1}^{\infty} g_n(x)$ is absolutely and uniformly convergent.

Note that $|a_n \cos n\theta| \le |a_n|$, $|b_n \sin n\theta| \le |b_n|$, $|c_n e^{in\theta}| \le |c_n|$

moreover $|c_{\pm n}| \le |a_n| + |b_n|$, $|a_n| \le |c_n| + |c_{-n}|$, $|b_n| \le |c_n| + |c_{-n}|$

A Fourier series with coefficients a_n, b_n or c_n converges uniformly and absolutely if

i) $\sum_{n=1}^{\infty} |a_n| < \infty$ & $\sum_{n=1}^{\infty} |b_n| < \infty$ or equivalently

ii) $\sum_{n=-\infty}^{\infty} |c_n| < \infty$.

The Fourier series of

$$f(\theta) = \begin{cases} \pi - \theta & 0 \le \theta \le \pi \\ \pi + \theta & -\pi \le \theta \le 0 \end{cases}$$

is

$$\pi/2 + \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos(2k+1)\theta}{(2k+1)^2}$$

$\left| \frac{\cos(2k+1)\theta}{(2k+1)^2} \right| \le \frac{1}{(2k+1)^2} = M_k$

$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} < \infty$

uniformly & absolutely

the one for

$$f(\theta) = \begin{cases} -1 & -\pi < \theta < 0 \\ 1 & 0 < \theta < \pi \end{cases}$$

is $\frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin(2k+1)\theta}{2k+1}$ converges pointwise

$\left| \frac{\sin(2k+1)\theta}{(2k+1)} \right| \le \frac{1}{2k+1}$

$\sum_{k=0}^{\infty} \frac{1}{2k+1} = +\infty$

Q: How do you justify what you saw in Desmos?

Theorem: If f is 2π-periodic, continuous, and piecewise smooth then the Fourier series of f converges uniformly and absolutely on $\mathbb{R}$.

Proof Since f is continuous and piecewise smooth its Fourier series converges pointwise to f, let c_n be the corresponding coefficients. In this case the Fourier series of f' with coefficients c_n' converges pointwise. Recall

$$c_n' = in\, c_n \quad \text{thus} \quad c_n = \frac{1}{in} c_n'$$

Bessel's inequality applied to f' yields

$$\sum_{n=-\infty}^{\infty} |c_n'|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f'(\theta)|^2 \, d\theta$$

Hence

$$\sum_{n=-N}^{N} |c_n| = |c_0| + \sum_{\substack{n=-N \\ n \neq 0}}^{N} \frac{|c_n'|}{n}$$

$$\underbrace{\sum_{n=-N}^{N} |c_n|}_{\text{Recall}} = |c_0| + \underbrace{\sum_{\substack{n=-N \\ n\neq 0}}^{N} \frac{|c_n'|}{n}}_{\uparrow} \leq |c_0| + \underbrace{\left(\sum_{\substack{n=-N \\ n\neq 0}}^{N} |c_n'|^2\right)^{1/2}}_{\text{by Bessel's} < \infty} \underbrace{\left(\sum_{\substack{n=-N \\ n\neq 0}}^{-N} \frac{1}{n^2}\right)^{1/2}}_{< \infty}$$

linear algebra $\quad v = (v_1, \dots v_N)$

$\quad u = (u_1, \dots u_N)$

$$\langle u, v \rangle = \sum_{j=1}^{N} v_j u_j$$

$\uparrow$ inner (dot) product

$$|\langle u, v \rangle| \leq \|u\| \, \|v\|$$

$\uparrow$ Cauchy-Schwarz

here $\|u\| = \left(\sum_{i=1}^{N} u_i^2\right)^{1/2}$

Cauchy - Schwarz

$$\sum_{n=-N}^{N} |c_n| = |c_0| + \sum_{\substack{n=-N \\ n\neq 0}}^{N} \frac{|c_n'|}{n} \leq |c_0| + \left(\sum_{\substack{n\neq 0 \\ n=-N}}^{N} |c_n'|^2 \right)^{1/2} \left(\sum_{\substack{n\neq 0 \\ n=-N}}^{N} \frac{1}{n^2} \right)^{1/2}$$

for every N

$$\sum_{n=-N}^{N} |c_n| \leq |c_0| + \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f'(\theta)|^2 \, d\theta \right)^{1/2} \left(\sum_{\substack{-\infty \\ n\neq 0}}^{\infty} \frac{1}{n^2} \right)^{1/2}$$

$N \longrightarrow \infty$

$$\sum_{n=-\infty}^{\infty} |c_n| < \infty$$

∎

Remark : If $f \in PC(\mathbb{R})$ we redefine f at the points of discontinuity as

$$\frac{f(\theta-) + f(\theta+)}{2}$$

Corollary 1 : If f is 2π periodic and piecewise smooth on $\mathbb{R}$ $S_N^f \to f$ as $N \to \infty$.

Corollary 2 : If f and g are 2π-periodic and piecewise smooth on $\mathbb{R}$ and f and g have the same Fourier coefficients, then $f = g$.

Pf : $S_N^f(\theta) = S_N^g(\theta)$ then letting $N \to \infty$ & Corollary 1 then $f(\theta) = g(\theta)$ modulo the redifinition above.