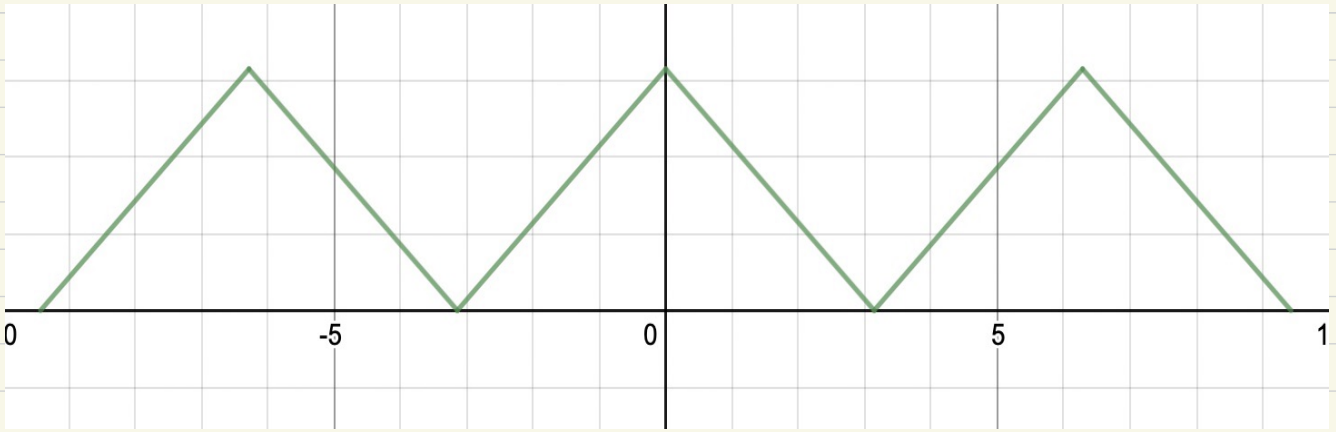
The Fourier series for

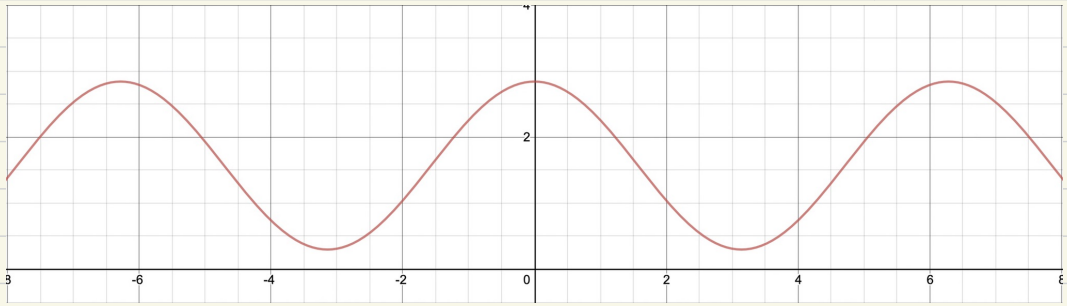
$$f(\theta) = \begin{cases} \pi - \theta & 0 \le \theta \le \pi \\ \pi + \theta & -\pi \le \theta \le 0 \end{cases}$$



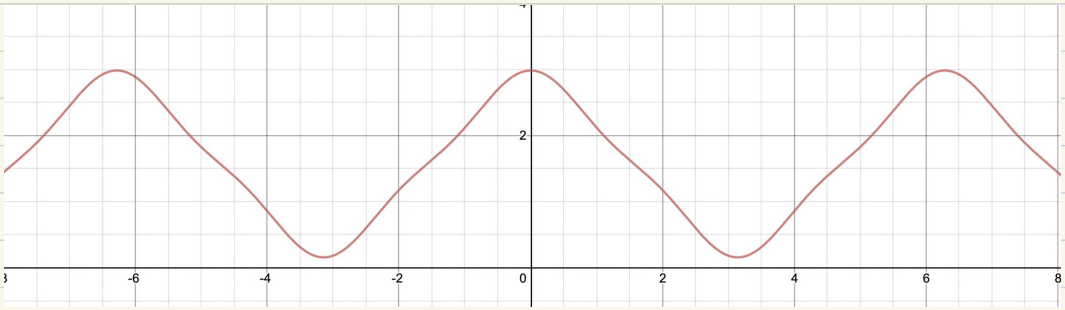
is

$$\pi/2 + 4/\pi \sum_{k=0}^{\infty} \frac{\cos(2k+1)\theta}{(2k+1)^2}$$

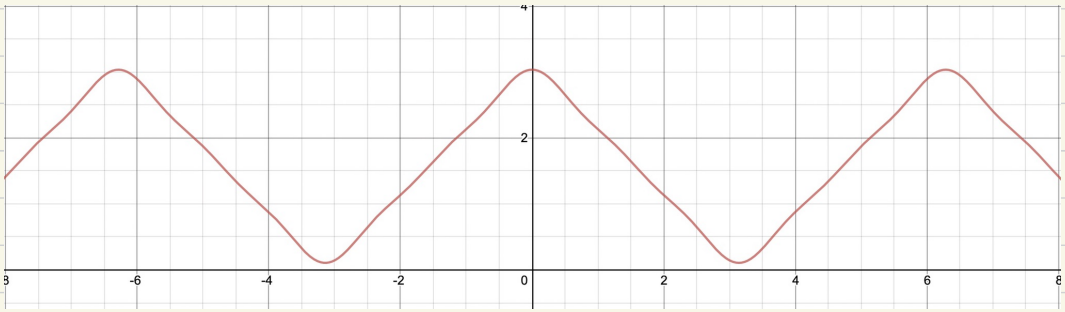
Let $S_m = \frac{\pi}{2} + 4/\pi \sum_{k=0}^{m} \frac{\cos(2k+1)\theta}{(2k+1)^2}$. Using Desmos graph S_1, S_2, S_3 and S_4. What do you observe?



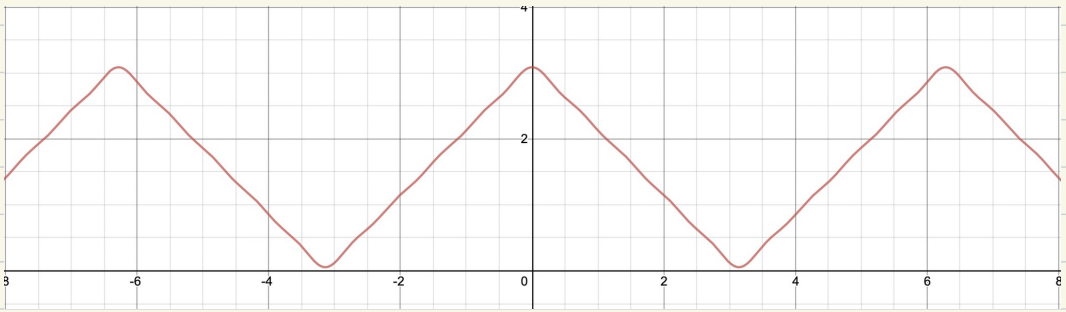
S_1



S_2



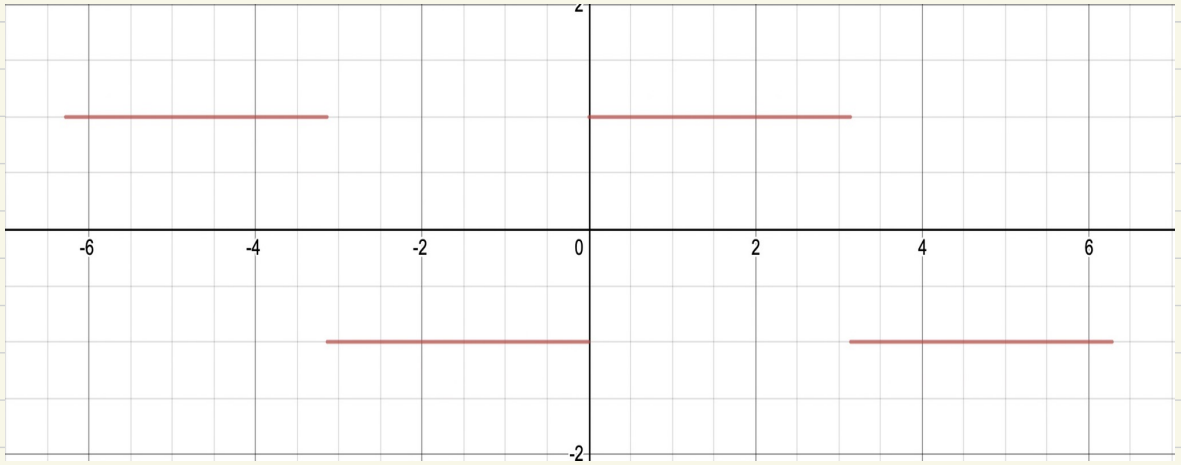
S_3



S_4

The Fourier series for

$$f(\theta) = \begin{cases} -1 & -\pi < \theta < 0 \\ 1 & 0 < \theta < \pi \end{cases}$$

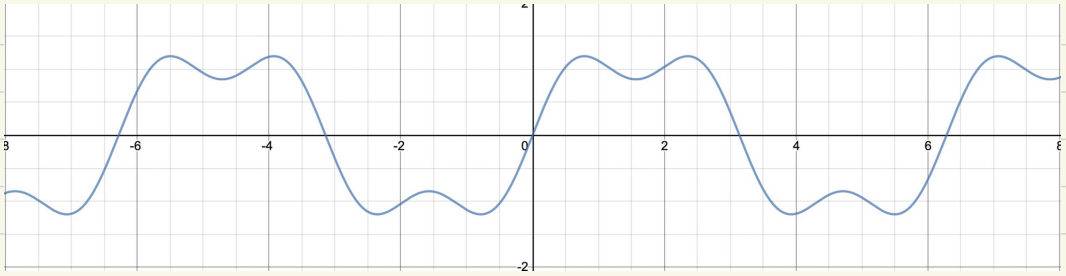


is

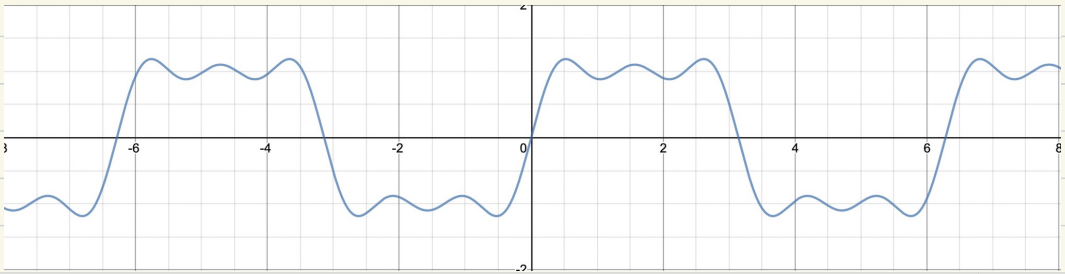
$$\frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin(2k+1)\theta}{2k+1}$$

Let $S_m = \frac{4}{\pi} \sum_{k=0}^{m} \frac{\sin(2k+1)\theta}{(2k+1)}$. Using Desmos graph

S_1, S_2, S_3 and S_4. What do you observe?



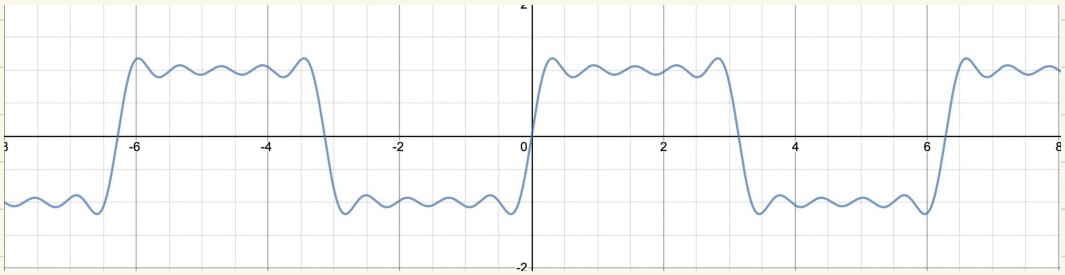
S_1



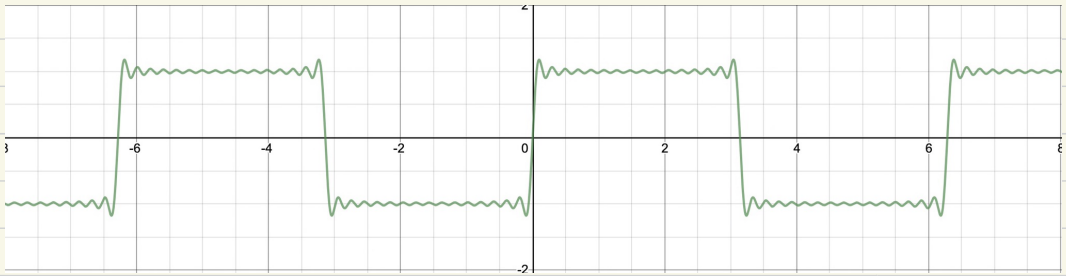
S_2



S_3



S_4



S_{15}

Question: Does the Fourier series of a periodic function f converge to f?

Answer: yes provided the convergence is interpreted suitably

Suppose $-\infty < a < b < +\infty$ - We say that $f : [a,b] \to \mathbb{R}$ is piecewise continuous provided that:

(1) f is continuous on $[a,b]$ except perhaps at finitely many points $x_1, \dots, x_k$.

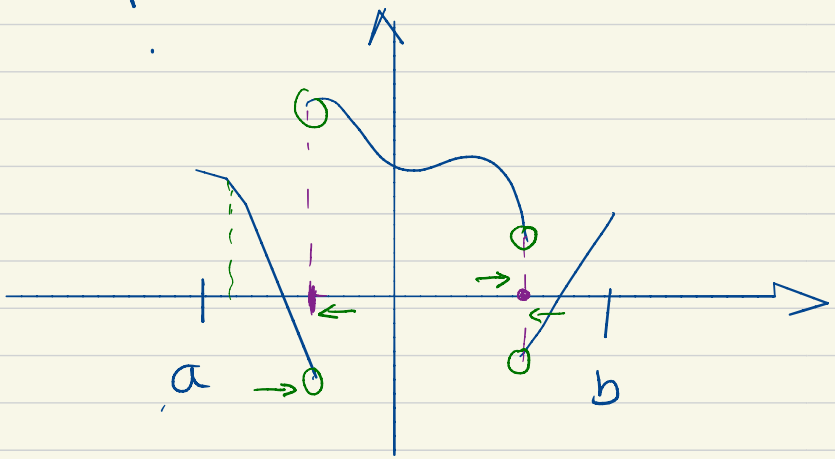
(2) at each of the points $x_1, \dots x_k$ the left-hand and right-hand limits of f

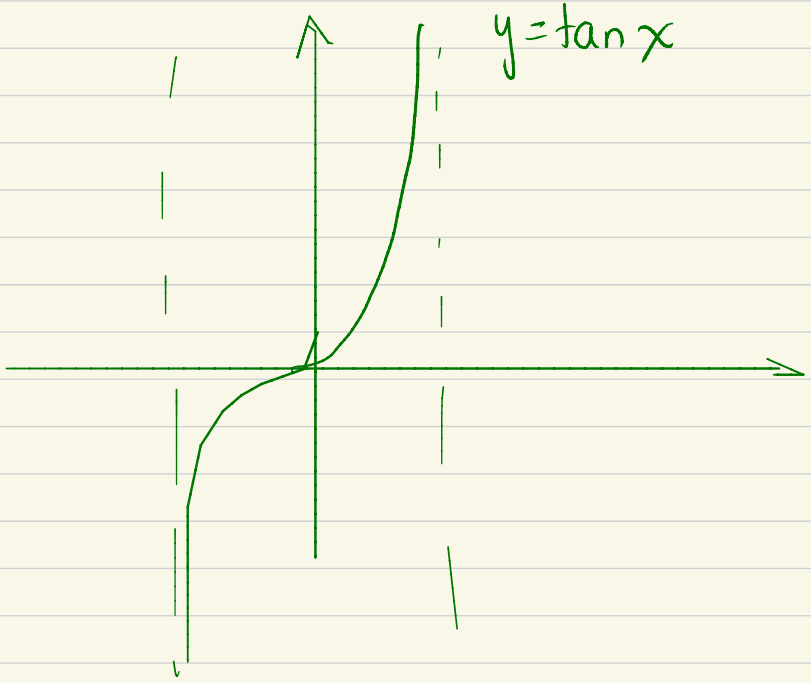
$$f(x_j-) = \lim_{h \to 0^+} f(x_j - h) \quad \& \quad f(x_j+) = \lim_{h \to 0^+} f(x_j + h) \text{ exist.}$$

$(h \to 0 ; h > 0)$ If a or b are one of the exceptional points then only $f(a+)$ or $f(b-)$ are required to exist.

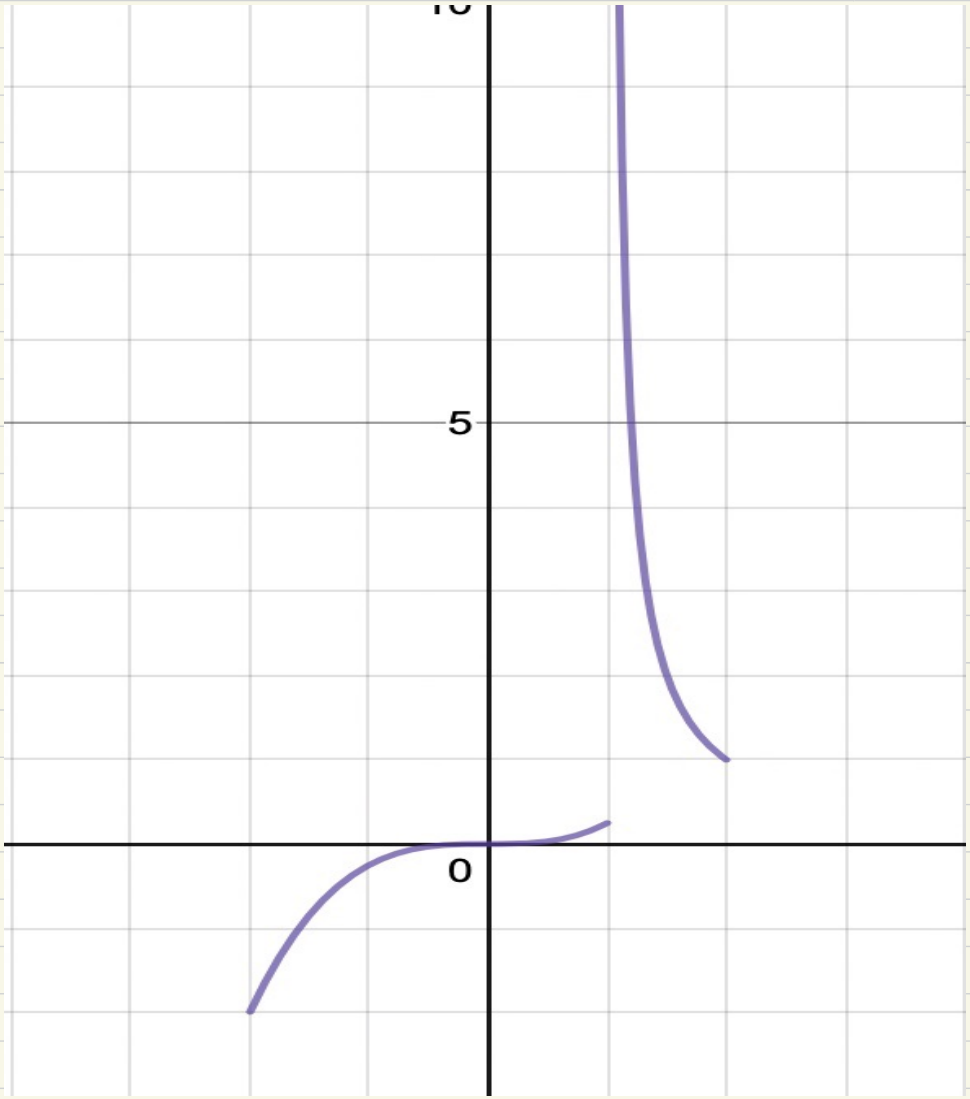
- f is piecewise continuous on $[a,b]$ if f is continuous there except for finitely many jump discontinuities, $f \in PC(a,b)$
- f on $[a,b]$ is piecewise smooth if f and f' are both piecewise continuous on $[a,b]$, $f \in PS(a,b)$

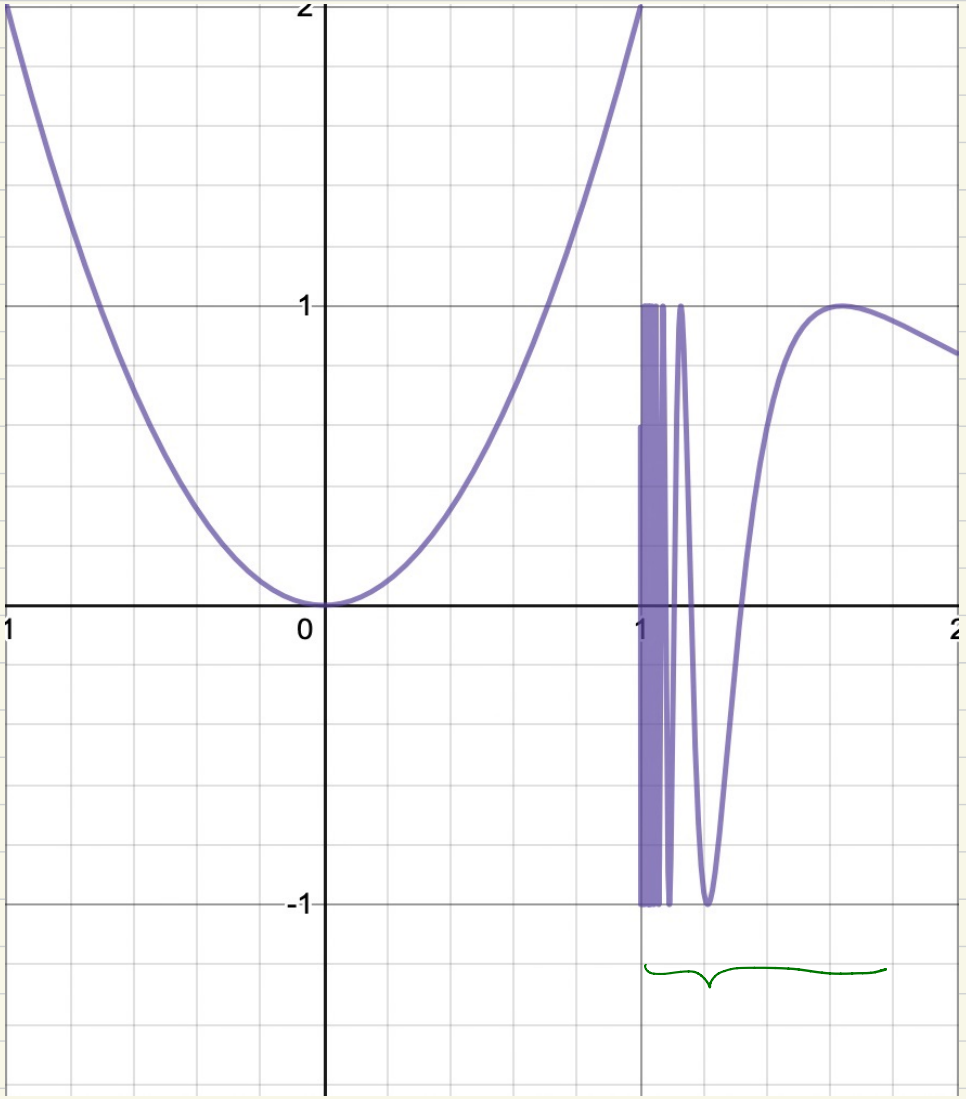
Examples:





Not piecewise continuous. Why?





$\sin \frac{1}{x-1}$

- $PC(\mathbb{R})$ is the set of piecewise continuous functions in $\mathbb{R}$, that is $f \in PC(\mathbb{R})$ if and only if $f \in PC[a,b]$ for all $-\infty < a < b < \infty$.

- $PS(\mathbb{R})$ is the set of piecewise smooth functions in $\mathbb{R}$, that is $f \in PS(\mathbb{R})$ if and only if $f \in PS[a,b]$ for all $-\infty < a < b < \infty$.

- Recall that an infinite series is the limit of its partial sums. If

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

is the Fourier series of f, a 2π-periodic function, the partial sum

$$S_N^f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{N} (a_n \cos n\theta + b_n \sin n\theta) = \sum_{n=-N}^{N} c_n e^{in\theta}$$

$$S_N^f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{N}(a_n \cos n\theta + b_n \sin n\theta) = \sum_{n=-N}^{N} c_n e^{in\theta}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\psi)\cos n\psi \, d\psi \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\psi)\sin n\psi \, d\psi$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\psi)e^{-in\psi} d\psi$$

$$S_N^f(\theta) = \frac{1}{2\pi}\sum_{n=-N}^{N}\left(\int_{-\pi}^{\pi} f(\psi)e^{-in\psi} d\psi\right)e^{in\theta} = \frac{1}{2\pi}\sum_{n=-N}^{N}\int_{-\pi}^{\pi} f(\psi)e^{in(\theta-\psi)} d\psi$$

replace n with $-n$

$$= \frac{1}{2\pi}\sum_{n=-N}^{N}\int_{-\pi}^{\pi} f(\psi)e^{in(\psi-\theta)} d\psi = \frac{1}{2\pi}\sum_{n=-N}^{N}\int_{-\pi+\theta}^{\pi+\theta} f(\theta+\phi)e^{in\phi} d\phi$$

$$\boxed{\phi = \psi - \theta}$$

since f is 2π-periodic

$$= \frac{1}{2\pi}\sum_{n=-N}^{N}\int_{-\pi}^{\pi} f(\theta+\phi)e^{in\phi} d\phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta+\phi)\left(\sum_{n=-N}^{N} e^{in\phi}\right) d\phi$$

$$S_N^f(\theta) = \int_{-\pi}^{\pi} f(\theta+\phi)\, D_N(\phi)\, d\phi \quad \text{where} \quad \boxed{D_N(\phi) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{in\phi}}$$

$$D_N(\phi) = \frac{1}{2\pi} e^{-iN\phi} \sum_{n=0}^{2N} e^{in\phi} = \frac{1}{2\pi} e^{-iN\phi} \frac{e^{i(2N+1)\phi} - 1}{e^{i\phi} - 1}$$

$$\left(e^{i\phi}\right)^n$$

because $\sum_{n=0}^{m} r^n = \frac{r^{m+1} - 1}{r - 1}$

$$\sum_{n=-N}^{N} e^{in\phi} = e^{-iN\phi} + e^{-i(N-1)\phi} + \cdots + e^{-i\phi} + 1 + e^{i\phi} + \cdots e^{iN\phi}$$

$$= e^{-iN\phi}\left(1 + e^{i\phi} + \cdots + e^{i(N-1)\phi} + e^{iN\phi} + e^{i2N\phi} + \cdots e\right)$$

$$\frac{1}{2\pi} e^{-iN\phi} \frac{e^{i(2N+1)\phi} - 1}{e^{i\phi} - 1} = \frac{1}{2\pi} \frac{e^{i(N+1)\phi} - e^{-iN\phi}}{e^{i\phi} - 1} =$$

$$= \frac{1}{2\pi} \frac{e^{i\phi/2} \left(e^{i(N+1/2)\phi} - e^{-i(N+1/2)\phi} \right)}{e^{i\phi/2} \left(e^{i\phi/2} - e^{-i\phi/2} \right)} = \frac{1}{2\pi} \frac{2i \sin\left(N+\frac{1}{2}\right)\phi}{2i \sin \phi/2}$$

$$= \frac{1}{2\pi} \frac{\sin (N+1/2)\phi}{\sin \phi/2} = D_N(\phi)$$

$D_N(\phi) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{in\phi}$ is the N- Dirichlet kernel

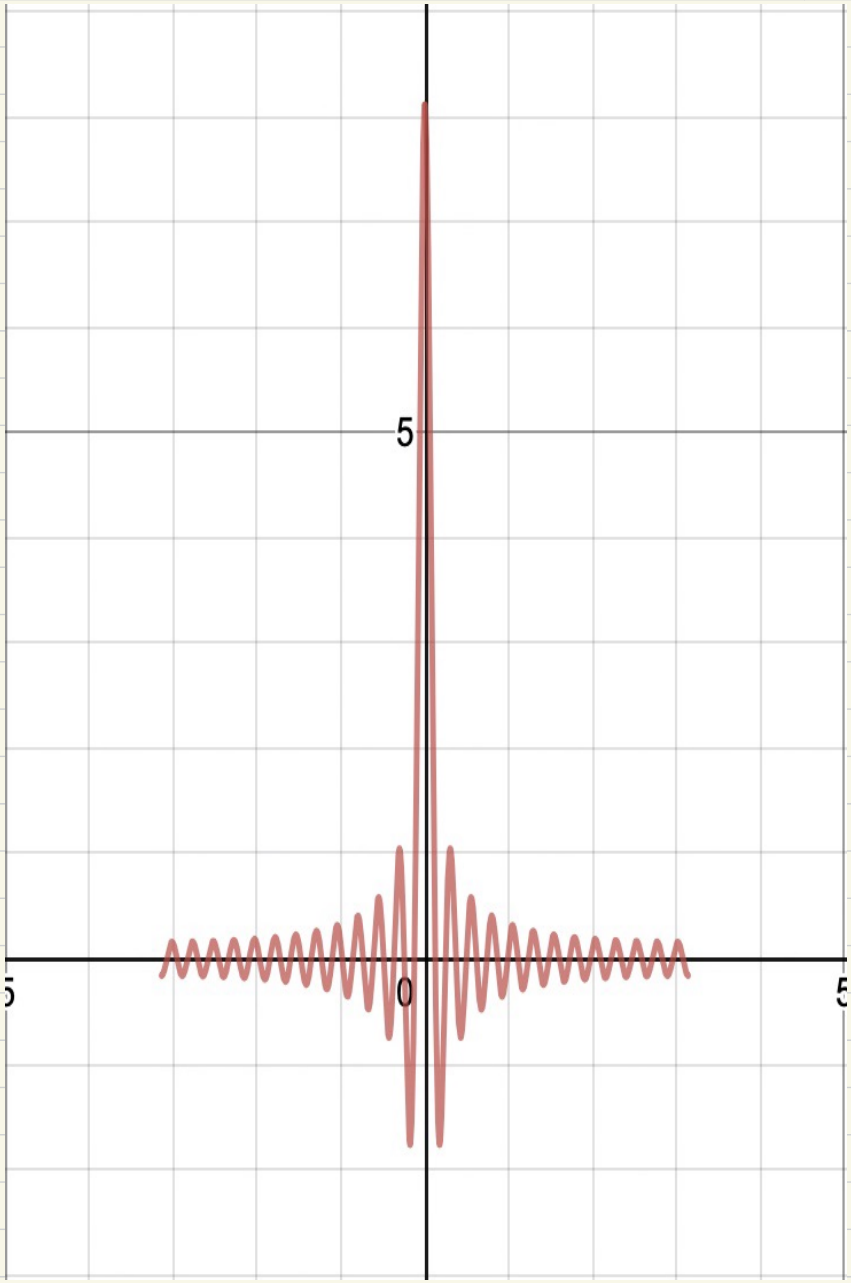
$$D_N(\phi) = \frac{1}{2\pi} e^{-iN\phi} \sum_{n=0}^{2N} e^{in\phi} = \frac{1}{2\pi} e^{-iN\phi} \left(1 + e^{i\phi} + \cdots + e^{i2N\phi}\right)$$

$$= \frac{1}{2\pi} e^{-iN\phi} \frac{e^{i(2N+1)\phi} - 1}{e^{i\phi} - 1} = \frac{1}{2\pi} \frac{e^{i(N+1)\phi} - e^{-iN\phi}}{e^{i\phi} - 1}$$

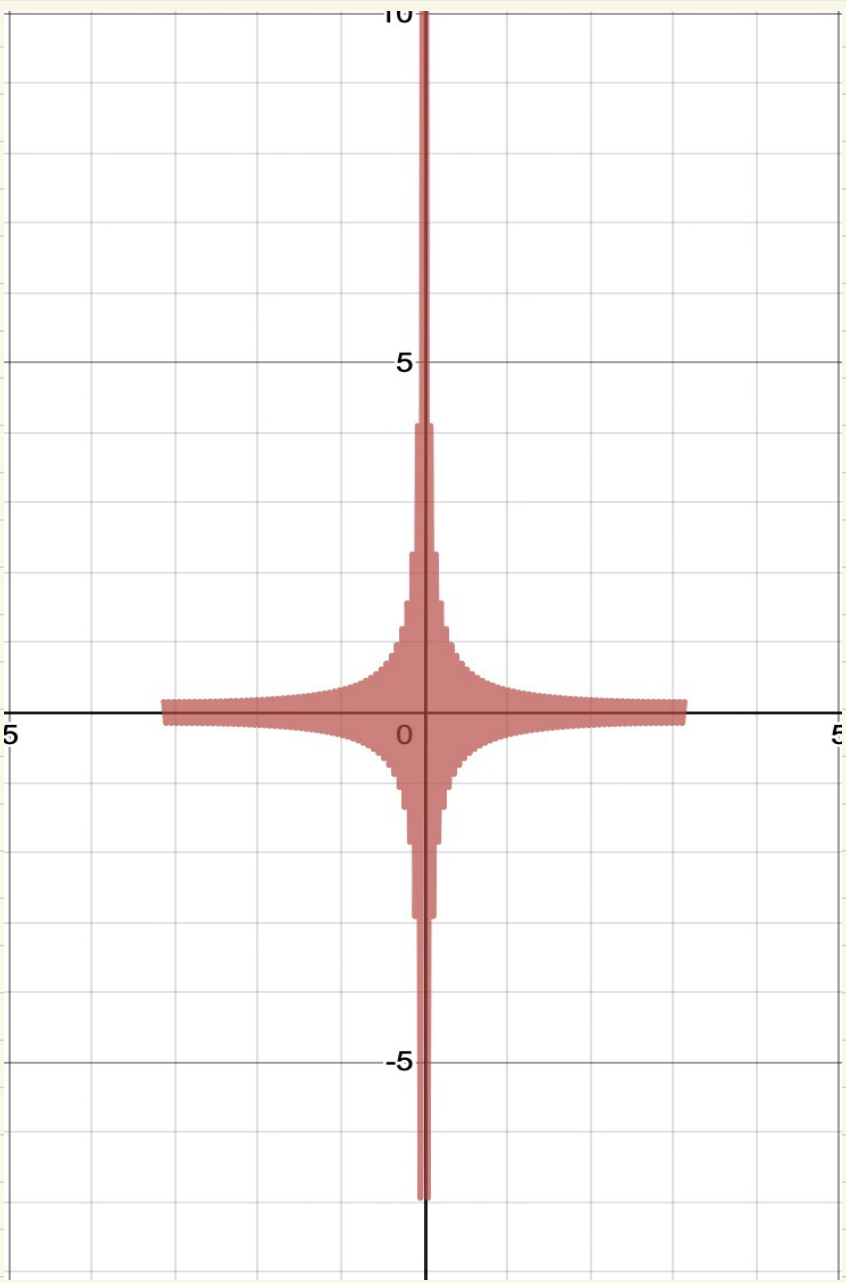
$$= \frac{1}{2\pi} \frac{e^{i\phi/2} \left(e^{i(N+\frac{1}{2})\phi} - e^{-i(N+\frac{1}{2})\phi}\right)}{e^{i\phi/2} \left(e^{i\phi/2} - e^{-i\phi/2}\right)}$$

$$= \frac{1}{2\pi} \frac{2i \sin\left(N+\frac{1}{2}\right)\phi}{2i \sin \phi/2} \Rightarrow \boxed{D_N(\phi) = \frac{1}{2\pi} \frac{\sin\left(N+\frac{1}{2}\right)\phi}{\sin \phi/2}}$$

$$\boxed{S_N^f(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta+\phi) \frac{\sin\left(N+\frac{1}{2}\right)\phi}{\sin\frac{\phi}{2}} \, d\phi}$$



$D_{25}(\phi)$



$D_{100}(\phi)$

Question: What is the intuition?

Lemma for all N

$$\int_{-\pi}^{0} D_N(\phi)\,d\phi = \int_{0}^{\pi} D_N(\phi)\,d\phi = \frac{1}{2}\int_{-\pi}^{\pi} D_N(\phi)\,d\phi = \frac{1}{2}$$

Proof: $D_N(-\phi) = D_N(\phi)$; ie D_N even

$$D_N(\phi) = \frac{1}{2\pi}\sum_{-N}^{N} e^{in\theta} = \frac{1}{2\pi} + \frac{1}{2\pi}\sum_{n=1}^{N}\left(e^{in\theta} + e^{-in\theta}\right)$$

$$= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{N}\cos n\phi$$

$$\int_0^{\pi} D_N(\phi)\,d\phi = \int_0^{\pi}\frac{1}{2\pi}\,d\phi + \frac{1}{\pi}\int_0^{\pi}\left(\sum_{n=1}^{N}\cos n\phi\right)d\phi$$

$$\int_0^{\pi} D_N(\phi)\,d\phi = \int_0^{\pi} \frac{1}{2\pi}\,d\phi + \frac{1}{\pi}\int_0^{\pi}\left(\sum_{n=1}^{N}\cos n\phi\right)d\phi$$

$$= \frac{\phi}{2\pi}\Big|_0^{\pi} + \frac{1}{\pi}\sum_{n=1}^{N}\int_0^{\pi}\cos n\phi\,d\phi = \boxed{\frac{1}{2}} + \frac{1}{\pi}\sum_{n=1}^{N}\underbrace{\frac{\sin n\phi}{n}\Big|_0^{\pi}}_{0}$$

Theorem: If f is 2π-periodic and piecewise smooth on $\mathbb{R}$ then

$$\lim_{N\to\infty} S_N^f(\theta) = \frac{1}{2}\left[f(\theta-) + f(\theta+)\right]$$

R - problems

Problem 5 § 1.3 : By separation of variables, derive the solutions

$u_n(x,y) = \sin(n\pi x)\sinh(n\pi y)$ of $u_{xx} + u_{yy} = 0$ $u(0,y) = u(1,y) = u(x,0) = 0$

Problems 4, 5, 9 § 2.1 Show that these are the Fourier series for the corresponding functions

$f(\theta) = \begin{cases} 0 & -\pi < \theta < 0 \\ \theta & 0 < \theta < \pi \end{cases}$ $\frac{1}{2}(f_1 + f_2)$	$\frac{\pi}{4} - \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{\cos(2n+1)\theta}{(2n+1)^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin n\theta$
$f(\theta) = \sin^2\theta$	$\frac{1}{2} - \frac{1}{2}\cos 2\theta$
$f(\theta) = \lvert\cos\theta\rvert$	$\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2n\theta}{4n^2 - 1}(-1)^n$

How to get problem 9 from problem 8 -

The Fourier series of $|\sin\theta|$ is $\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2n\theta}{4n^2-1}$

$$\sin(\theta + \pi/2) = \sin\theta\cos\pi/2 + \cos\theta\sin\pi/2 = \cos\theta$$

Hence the Fourier series of $|\cos\theta|$ is the Fourier series above applied to $\theta + \pi/2$ that is

$$2/\pi - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2n(\theta+\pi/2)}{4n^2-1} = 2/\pi - 4/\pi \sum_{n=1}^{\infty} \frac{(-1)^n \cos 2n\theta}{4n^2-1}$$

$$\cos 2n(\theta+\pi/2) = \cos(2n\theta + n\pi) = \cos 2n\theta \cos n\pi - \sin 2n\theta \sin n\pi$$

($\sin n\pi = 0$)

$$= (-1)^n \cos 2n\theta$$

Problem 5 § 1.3 : By separation of variables, derive the solutions

$u_n(x,y) = \sin(n\pi x)\sinh(n\pi y)$ of $u_{xx}+u_{yy}=0$ $u(0,y)=u(1,y)=u(x,0)=0$

$u_n(x,y) = X(x)Y(y)$ $\quad u_{xx}+u_{yy} = X''(x)Y(y) + X(x)Y''(y) = 0$

assume $X(x)\neq 0$ & $Y(y)\neq 0$ then $\dfrac{X''(x)}{X(x)} = -\dfrac{Y''(y)}{Y(y)} = A$ constant

w.l.o.g $A = -\alpha^2$ then

$$\begin{cases} X''(x) + \alpha^2 X(x) = 0 \\ X(0) = X(1) = 0 \end{cases}$$

$$X(x) = C_1\cos\alpha x + C_2\sin\alpha x$$

$X(0) = C_1 = 0$

$X(1) = C_2\sin\alpha = 0$

$\alpha = n\pi$

$$\boxed{X(x) = C_1\sin n\pi x}$$

$$\begin{cases} Y''(y) - \alpha^2 Y(y) = 0 \\ Y(0) = 0 \end{cases}$$

$$Y(y) = C_1\cosh\alpha y + C_2\sinh\alpha y$$

$Y(0) = C_1 = 0$

$$\boxed{Y(y) = C_2\sinh(n\pi y)}$$

$$\boxed{u_n(x,y) = \sin(n\pi x)\sinh(n\pi y)}$$

Problem 4 §2.1 The Fourier series of $f(\theta) = \begin{cases} 0 & -\pi < \theta < 0 \\ \theta & 0 < \theta < \pi \end{cases}$ is

$$\frac{\pi}{4} - \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{\cos(2n+1)\theta}{(2n+1)^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\theta$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \theta \cos n\theta \, d\theta$$

$n = 0 \qquad a_0 = \frac{1}{\pi}\frac{\theta^2}{2}\Big|_0^{\pi} = \frac{\pi}{2}$

$$\int_0^{\pi} \theta \cos n\theta \, d\theta = \frac{\theta \sin n\theta}{n}\Big|_0^{\pi} - \int_0^{\pi} \frac{\sin n\theta}{n} \, d\theta$$

$$= + \frac{\cos n\theta}{n^2}\Big|_0^{\pi} = \begin{cases} 0 & n \text{ even} \\ -\frac{2}{n^2} & n \text{ odd} \end{cases}$$

$$a_n = -\frac{2}{\pi}\frac{\cos(2n+1)\theta}{(2n+1)^2}$$

$$\frac{1}{2}a_0 = \frac{\pi}{4}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \theta \sin n\theta \, d\theta$$

$b_0 = 0$

$$\int_0^{\pi} \theta \sin n\theta \, d\theta = -\frac{\theta \cos n\theta}{n}\Big|_0^{\pi} + \int_0^{\pi} \frac{\cos n\theta}{n} \, d\theta = -\frac{\pi}{n}(-1)^n$$

$$+ \frac{\sin n\theta}{n^2}\Big|_0^{\pi} = \frac{\pi}{n}(-1)^{n+1}$$

$$b_n = \frac{\pi}{n}(-1)^{n+1}$$

Problem 5 § 2.1 the Fourier series of $f(\theta) = \sin^2\theta$ is

$$\frac{1}{2} - \frac{1}{2}\cos 2\theta -$$

Recall the double angle formula $\cos 2\theta = 1 - 2\sin^2\theta$ thus

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

Problem 9 § 2.1 The Fourier series of $f(\theta) = |\cos\theta|$ is

$$\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n \cos 2n\theta}{4n^2 - 1}$$

Note $f(-\theta) = f(\theta)$, f even; $b_n = 0$ and

$$a_n = \frac{2}{\pi}\int_0^{\pi} |\cos\theta| \cos n\theta \, d\theta = \frac{2}{\pi}\int_0^{\pi/2} \cos\theta \cos n\theta \, d\theta - \frac{2}{\pi}\int_{\pi/2}^{\pi} \cos\theta \cos n\theta \, d\theta$$

$$\cos\theta \cos n\theta = \frac{1}{2}\left[\cos(n+1)\theta + \cos(n-1)\theta\right]$$

$$\int \cos\theta \cos n\theta \, d\theta = \frac{1}{2}\int \left(\cos(n+1)\theta + \cos(n-1)\theta\right) d\theta \qquad \text{if } n \neq 1, 0$$

$$= \frac{1}{2}\left(\frac{\sin(n+1)\theta}{n+1} + \frac{\sin(n-1)\theta}{n-1}\right) + C$$

$n = 1$

$$\int \cos^2\theta \, d\theta = \int\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta = \frac{\theta}{2} + \frac{1}{4}\sin 2\theta + C$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} |\cos\theta|\, d\theta = \frac{2}{\pi}\int_0^{\pi/2} \cos\theta\, d\theta - \frac{2}{\pi}\int_{\pi/2}^{\pi} \cos\theta\, d\theta = \frac{2}{\pi}\sin\theta\Big|_0^{\pi/2} - \frac{2}{\pi}\sin\theta\Big|_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}(1+1) = \boxed{\frac{4}{\pi} = a_0}$$

$$a_1 = \frac{2}{\pi}\left(\theta/2 + \frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi/2} - \frac{2}{\pi}\left(\theta/2 + \frac{1}{4}\sin 2\theta\right)\Big|_{\pi/2}^{\pi} = \frac{2}{\pi}\left(\frac{\pi}{4} - \left(\frac{\pi}{2} - \frac{\pi}{4}\right)\right) = 0$$

$$\boxed{a_1 = 0}$$

$$a_n = \frac{1}{\pi}\left[\left(\frac{\sin(n+1)\theta}{n+1} + \frac{\sin(n-1)\theta}{(n-1)}\right)\Big|_0^{\pi/2} - \left(\frac{\sin(n+1)\theta}{n+1} + \frac{\sin(n-1)\theta}{n-1}\right)\Big|_{\pi/2}^{\pi}\right]$$

$$a_n = \frac{2}{\pi}\left[\frac{\sin(n+1)\pi/2}{(n+1)} + \frac{\sin(n-1)\pi/2}{(n-1)}\right] = \frac{2}{\pi}\begin{cases} 0 & \text{if } n \text{ odd} \\ \dfrac{\sin(2k+1)\pi/2}{(2k+1)} + \dfrac{\sin(2k-1)\pi/2}{(2k-1)} & \text{if } n = 2k \end{cases}$$

$$a_{2k} = \frac{2}{\pi}\left[\frac{(-1)^k}{(2k+1)} + \frac{(-1)^{k-1}}{(2k-1)}\right] = \frac{2(-1)^k}{\pi}\left(\frac{1}{2k+1} - \frac{1}{2k-1}\right)$$

$$a_{2k} = \frac{2}{\pi}(-1)^k \frac{(-2)}{(2k+1)(2k-1)} = \frac{-4}{\pi}\frac{(-1)^k}{4k^2-1}$$