$f : \mathbb{R} \longrightarrow \mathbb{C}$ Riemann integrable on bounded intervals (for example f bounded and continuous except at most in finitely many points in each interval).

Assume f is periodic with period 2π.

GOAL: expand f as a series of the form

$$(1) \qquad f(\theta) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

$$(2) \qquad f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

The relationship between these 2 expansions is obtained using:

$$\cos n\theta = \frac{e^{in\theta} + e^{-in\theta}}{2} \quad \& \quad \sin n\theta = \frac{e^{in\theta} - e^{-in\theta}}{2i}$$

plus the fact that $\cos(-x) = \cos x$ & $\sin(-x) = -\sin x$

This yields to the following relationships between the coefficients

$$c_0 = \frac{1}{2} a_0 \quad , \quad c_n = \frac{1}{2}(a_n - ib_n) \quad , \quad c_{-n} = \frac{1}{2}(a_n + ib_n)$$

$$a_0 = 2c_0 \quad , \quad a_n = c_n + c_{-n} \quad , \quad b_n = i(c_n - c_{-n})$$

Definition: Suppose f is 2π-periodic and integrable over $[-\pi, \pi]$. The numbers

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta \, d\theta \quad \text{and} \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta \, d\theta$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, e^{-in\theta} \, d\theta$$

are the Fourier coefficients of f, and

$$\sum_{n=-\infty}^{\infty} c_n e^{in\theta} \quad \text{or} \quad \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

is called the Fourier series of f.

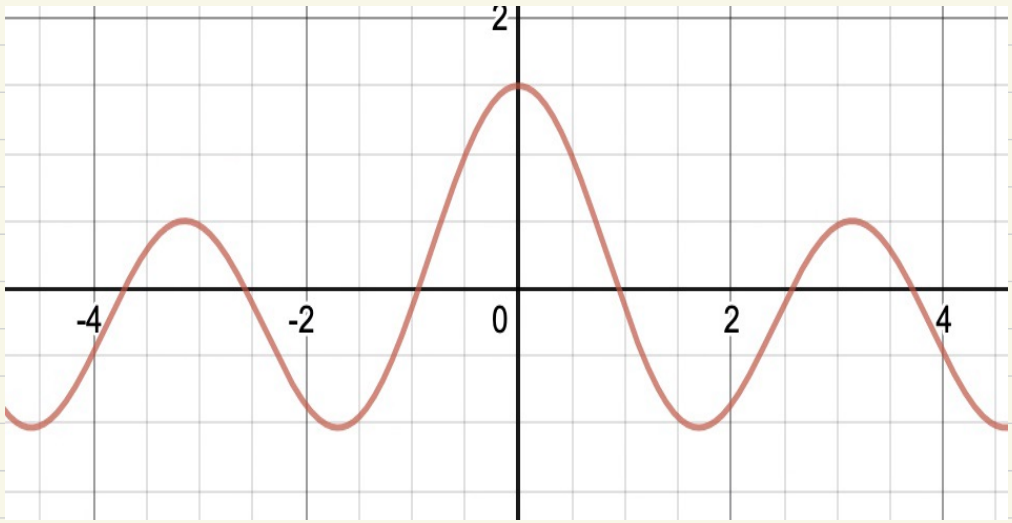
Note: $c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta$ average (mean value) of f in $[-\pi, \pi]$

Properties: ① If F is periodic of period P, then $\int_a^{a+P} F(x)\,dx$ is independent of a

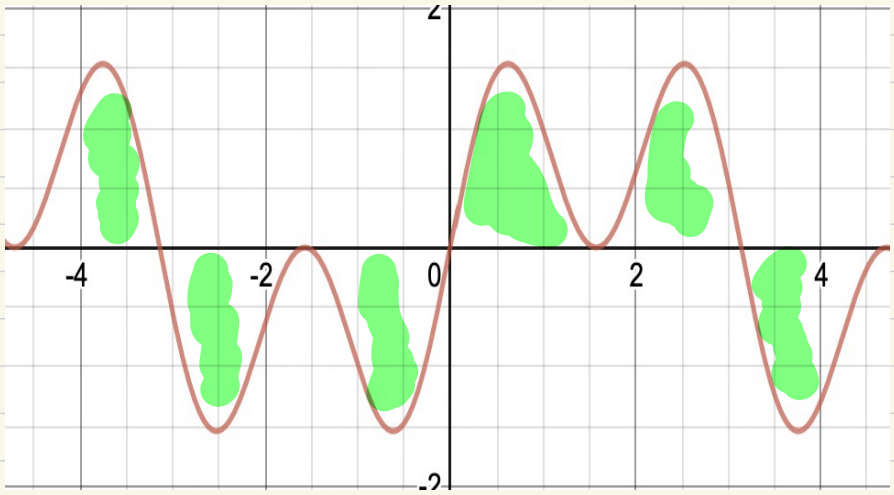
② Recall if $F(-x) = F(x)$, if $F(-x) = -F(x)$

F even F odd

$$\int_{-a}^{a} F(x)\,dx = \begin{cases} 2\int_0^a F(x)\,dx & \text{if } F \text{ even} \\ 0 & \text{if } F \text{ is odd} \end{cases}$$



F even



F odd

(3) If f even : $a_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos n\theta\, d\theta$ & $b_n = 0$

(4) If f odd : $a_n = 0$ & $b_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin n\theta\, d\theta$

Q: Why? If f even $f(\theta)\cos n\theta$ is also even

$$f(-\theta)\cos n(-\theta) = f(\theta)\cos(-n\theta) = f(\theta)\cos n\theta$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta = \frac{2}{\pi}\int_0^{\pi} \underbrace{f(\theta)\cos n\theta}_{\text{even function}}\, d\theta$$

$f(\theta)\sin n\theta$ is odd

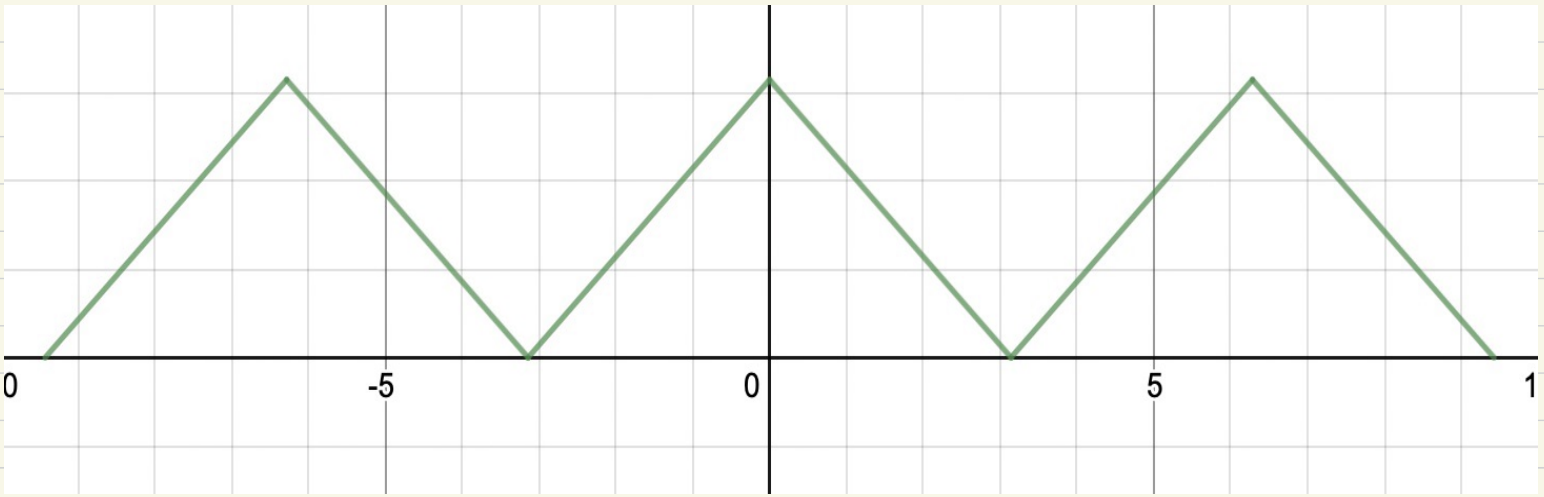
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta = 0$$

Compute the Fourier series for the following 2π-periodic functions

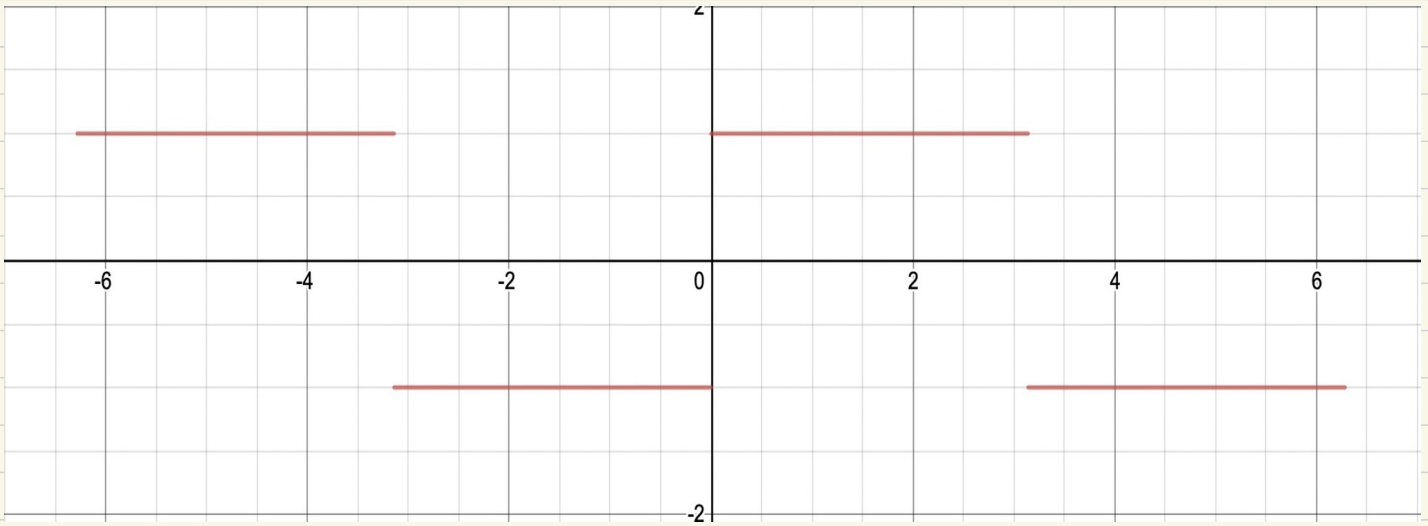
① $$f(\theta) = \begin{cases} \pi - \theta & 0 \le \theta \le \pi \\ \pi + \theta & -\pi \le \theta \le 0 \end{cases}$$

② $$f(\theta) = \begin{cases} -1 & -\pi < \theta < 0 \\ 1 & 0 < \theta < \pi \end{cases}$$

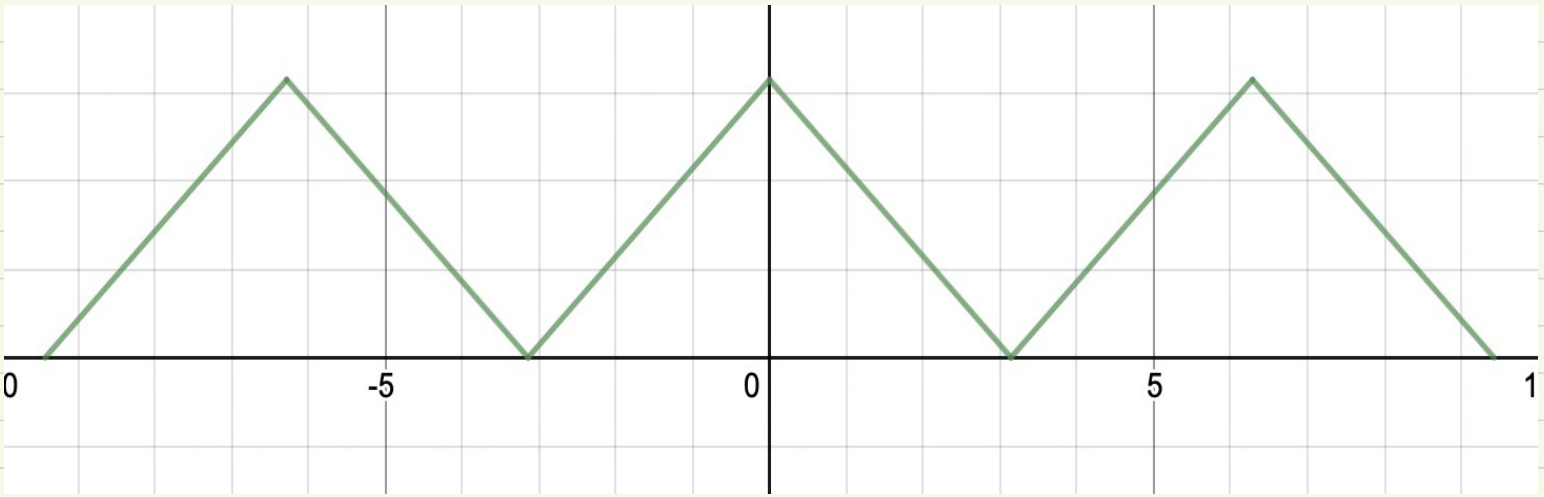
① $$f(\theta) = \begin{cases} \pi - \theta & 0 \le \theta \le \pi \\ \pi + \theta & -\pi \le \theta \le 0 \end{cases}$$



(2) $$f(\theta) = \begin{cases} -1 & -\pi < \theta < 0 \\ 1 & 0 < \theta < \pi \end{cases}$$



① $$f(\theta) = \begin{cases} \pi - \theta & 0 \le \theta \le \pi \\ \pi + \theta & -\pi \le \theta \le 0 \end{cases}$$



Properties : ① $f(-\theta) = f(\theta)$ f even

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(\theta) \cos n\theta \, d\theta \qquad n = 0, 1, 2$$

$b_n = 0$

$$\int_0^{\pi} f(\theta)\cos n\theta\, d\theta = \int_0^{\pi} (\pi-\theta)\cos n\theta\, d\theta$$

$$= \pi\int_0^{\pi}\cos n\theta\, d\theta - \int_0^{\pi}\theta\cos n\theta\, d\theta$$

If $\boxed{n=0}$ $\quad \pi\int_0^{\pi} d\theta - \int_0^{\pi}\theta\, d\theta = \pi\theta - \frac{\theta^2}{2}\Big|_0^{\pi} = \boxed{\frac{\pi^2}{2}}$

$n\neq 0$ $\quad \frac{\pi}{n}\underbrace{\sin n\theta}_{0}\Big|_0^{\pi} - \int_0^{\pi}\theta\cos n\theta\, d\theta$ integration by parts

$$= -\theta\underbrace{\frac{\sin n\theta}{n}}_{0}\Big|_0^{\pi} + \int_0^{\pi}\frac{\sin n\theta}{n}\, d\theta = -\frac{\cos n\theta}{n^2}\Big|_0^{\pi}$$

$n \neq 0$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(\theta) \cos n\theta \, d\theta = \frac{2}{\pi} \left(- \frac{\cos n\theta}{n^2} \Big|_0^{\pi} \right)$$

$$a_n = \frac{2}{\pi} \cdot \frac{-(-1)^n + 1}{n^2} = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{4}{\pi n^2} & \text{if } n \text{ odd} \end{cases}$$

$n = 0 \qquad a_0 = \frac{2}{\pi} \cdot \frac{\pi^2}{2} = \pi$

The Fourier series of f is

$$\frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1,3,5} \frac{\cos n\theta}{n^2}$$

$$= \pi/2 + 4/\pi \sum_{k=0}^{\infty} \frac{\cos (2k+1)\theta}{(2k+1)^2}$$

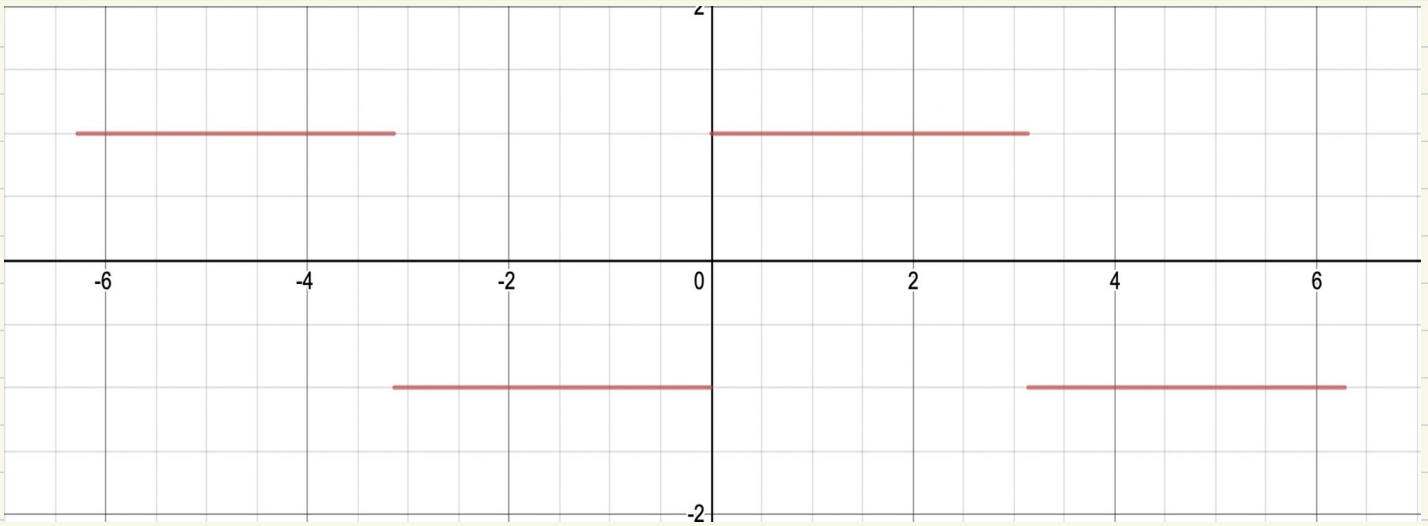
$$\sum_{k=0}^{\infty} \frac{\cos(2k+1)\theta}{(2k+1)^2}$$

$$\left|\frac{\cos(2k+1)\theta}{(2k+1)^2}\right| \leq \frac{1}{(2k+1)^2} \qquad \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} < \infty$$

$$\sum_{k=0}^{\infty} \frac{\sin(2k+1)\theta}{(2k+1)}$$

$$\left|\frac{\sin(2k+1)\theta}{(2k+1)}\right| \leq \frac{1}{2k+1} \qquad \sum_{k=0}^{\infty} \frac{1}{2k+1} = +\infty$$

(2) $$f(\theta) = \begin{cases} -1 & -\pi < \theta < 0 \\ 1 & 0 < \theta < \pi \end{cases}$$



Properties : $f(-\theta) = -f(\theta)$ f odd

$a_n = 0$ & $b_n = \frac{2}{\pi} \int_0^{\pi} f(\theta) \sin n\theta \, d\theta$ $n = 1, 2, 3$

$$\int_0^{\pi} f(\theta) \sin n\theta \, d\theta = \int_0^{\pi} \sin n\theta \, d\theta = -\frac{\cos n\theta}{n}\Big|_0^{\pi}$$

$$= -\frac{(-1)^n - 1}{n} = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{2}{n} & \text{if } n \text{ odd} \end{cases}$$

$$b_n = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{4}{\pi n} & \text{if } n \text{ is odd} \end{cases}$$

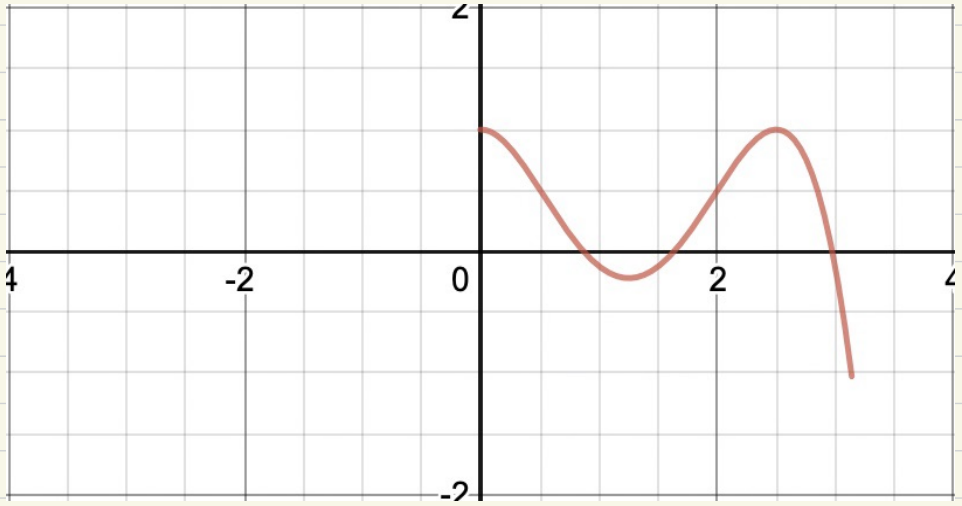
the Fourier series is $\frac{4}{\pi} \sum_{n=1,3,5} \frac{\sin n\theta}{n}$

$$\boxed{\frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin (2k+1)\theta}{2k+1}}$$

Note that what we did in examples (1) & (2) and what you are asked to do in the homework of § 2.1 is to extend a function f defined on $(-\pi, \pi]$ to a periodic function in $\mathbb{R}$ by setting

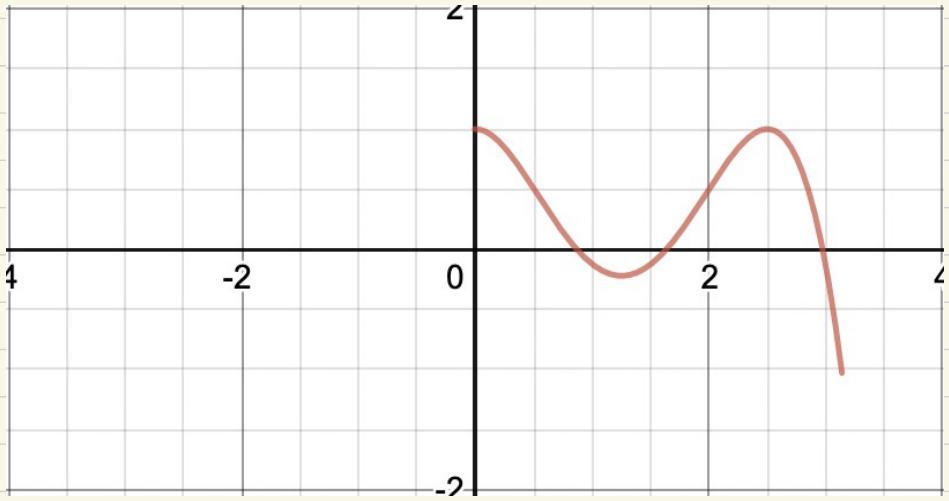
$$f(\theta + 2\pi n) = f(\theta) \quad \text{for all } \theta \in (-\pi, \pi] \text{ and integers } n.$$

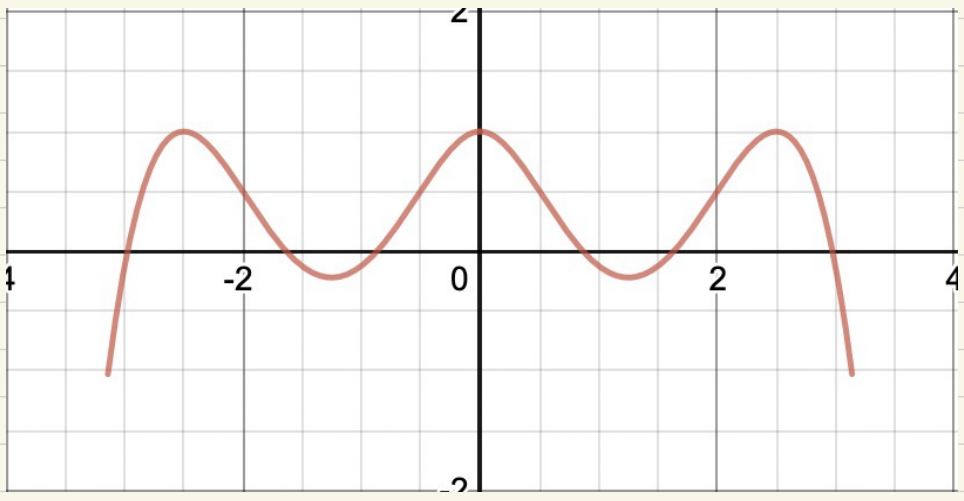
Assume f defined on $[0, \pi]$ and we want to build a 2π-periodic function:



Even extension

$$f_{even}(\theta) = \begin{cases} f(\theta) & \text{if } 0 \le \theta \le \pi \\ f(-\theta) & \text{if } -\pi \le \theta < 0 \end{cases}$$





$$\frac{1}{\pi}\int_{-\pi}^{\pi} f_{even}(\theta)\cos n\theta\, d\theta = \frac{2}{\pi}\int_{0}^{\pi} f(\theta)\cos n\theta\, d\theta = a_n$$

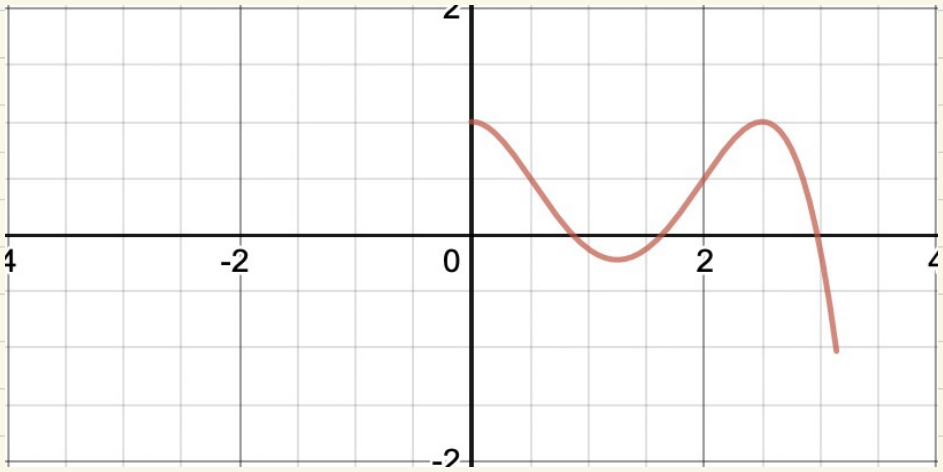
$$\int_{-\pi}^{\pi} f_{even}(\theta)\sin n\theta\, d\theta = 0$$

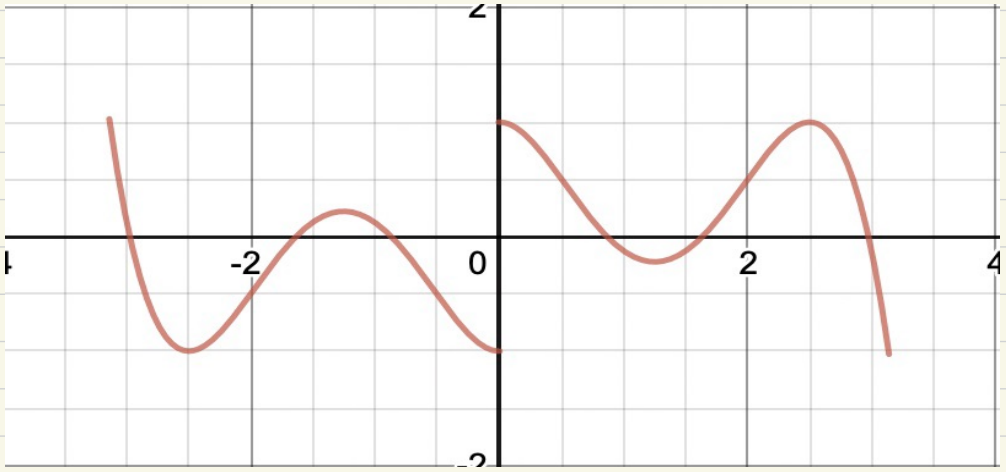
$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta$$

Fourier cosine series of f

Odd extension

$$f_{odd}(\theta) = \begin{cases} f(\theta) & \text{if } 0 < \theta < \pi \\ -f(-\theta) & \text{if } -\pi < \theta < 0 \\ 0 & \text{if } \theta = 0 \end{cases}$$





$$\int_{-\pi}^{\pi} f_{odd}(\theta) \cos n\theta \, d\theta = 0$$

$$\sum_{n=1}^{\infty} b_n \sin n\theta$$

Fourier sine series of f

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f_{odd}(\theta) \sin n\theta \, d\theta = \frac{2}{\pi}\int_{0}^{\pi} f(\theta) \sin n\theta \, d\theta = b_n$$

Bessel's inequality: If f is 2π-periodic and Riemann integrable on $[-\pi, \pi]$ with Fourier coefficients

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, e^{-in\theta}\, d\theta$$ then

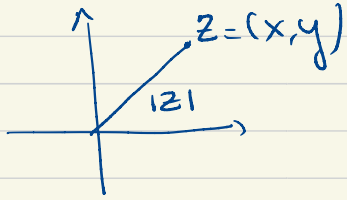
$$\sum_{-\infty}^{\infty} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2\, d\theta$$

Proof: Recall $z \in \mathbb{C}$ $z = x + iy$; $\bar{z} = x - iy$ (conjugate)

$$|z|^2 = z\bar{z} = (x+iy)(x-iy) = x^2 + y^2 \quad \left(e^{in\theta} = e^{-in\theta} \right)$$

Compute

$$\left| f(\theta) - \sum_{-N}^{N} c_n e^{in\theta} \right|^2 \geq 0$$



$$\int_{-\pi}^{\pi} \left| f(\theta) - \sum_{-N}^{N} c_n e^{in\theta} \right|^2 = \int_{-\pi}^{\pi} \left(f(\theta) - \sum_{-N}^{N} c_n e^{in\theta} \right) \left(\overline{f(\theta)} - \sum_{-N}^{N} \bar{c}_n e^{-in\theta} \right)$$

$$\int_{-\pi}^{\pi} f(\theta)\overline{f(\theta)} - \sum_{-N}^{N} f(\theta)\bar{c}_n e^{-in\theta} - \sum_{-N}^{N} c_n \overline{f(\theta)} e^{in\theta} + \underbrace{\sum_{n,k=-N}^{N} c_n \bar{c}_k e^{in\theta} e^{-ik\theta}}$$

$$\int_{-\pi}^{\pi} \left(|f(\theta)|^2 - \sum_{-N}^{N} \bar{c}_n f(\theta) e^{-in\theta} - \sum_{-N}^{N} c_n \overline{f(\theta)} e^{in\theta} + \right) \geq 0$$

$$\int_{-\pi}^{\pi} |f(\theta)|^2 \, d\theta \qquad + \int_{-\pi}^{\pi} \sum_{-N}^{N} c_n \bar{c}_k e^{in\theta - ik\theta} \, d\theta$$

$$\sum_{-N}^{N} c_n \bar{c}_k \int_{-\pi}^{\pi} e^{i(n-k)\theta} \, d\theta = \sum_{-N}^{N} c_n \bar{c}_k \begin{cases} 2\pi & n = k \\ 0 & \text{if } n \neq k \end{cases}$$

$$2\pi \sum_{-N}^{N} c_n \bar{c}_n = 2\pi \sum_{-N}^{N} |c_n|^2$$

$$\int_{-\pi}^{\pi} \left| f(\theta) - \sum_{-N}^{N} c_n e^{in\theta} \right|^2 d\theta = \int_{-\pi}^{\pi} |f(\theta)|^2 + 2\pi \sum_{-N}^{N} |c_n|^2$$

$$- \int_{-\pi}^{\pi} \left(\sum_{-N}^{N} \bar{c}_n f(\theta) e^{-in\theta} + \sum_{-N}^{N} c_n \bar{f}(\theta) e^{in\theta} \right) d\theta$$

$$\sum_{-N}^{N} \bar{c}_n \underbrace{\int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta}_{2\pi c_n} + \sum_{-N}^{N} c_n \underbrace{\int_{-\pi}^{\pi} \bar{f}(\theta) e^{in\theta} d\theta}_{\overline{\int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta} = 2\pi \bar{c}_n}$$

$$\underbrace{2\pi \sum_{-N}^{N} \bar{c}_n c_n + 2\pi \sum_{-N}^{N} c_n \bar{c}_n}_{4\pi \sum_{-N}^{N} |c_n|^2}$$

$$\int_{-\pi}^{\pi} \left| f(\theta) - \sum_{-N}^{N} c_n e^{in\theta} \right|^2 d\theta = \int_{-\pi}^{\pi} |f(\theta)|^2 + 2\pi \sum_{-N}^{N} |c_n|^2 - 4\pi \sum_{-N}^{N} |c_n|^2$$

$$= \int_{-\pi}^{\pi} |f(\theta)|^2 - 2\pi \sum_{-N}^{N} |c_n|^2 \geq 0$$

$$\sum_{-N}^{N} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2$$

$$\sum_{-\infty}^{\infty} |c_n|^2 \leq$$

$|c_n| \longrightarrow 0 \qquad n \to \infty$

$$\left| f(\theta) - \sum_{-N}^{N} c_n e^{in\theta} \right|^2 = \left(f(\theta) - \sum_{-N}^{N} c_n e^{in\theta} \right)\left(\overline{f(\theta)} - \sum_{-N}^{N} \overline{c_n} e^{-in\theta} \right)$$

$$= |f(\theta)|^2 - \sum_{-N}^{N} \left[f(\theta)\overline{c_n} e^{-in\theta} + \overline{f(\theta)} c_n e^{in\theta} \right] + \left| \sum_{-N}^{N} c_n e^{in\theta} \right|^2$$

$$\left| \sum_{-N}^{N} c_n e^{in\theta} \right|^2 = \sum_{n,m=-N}^{N} c_m \overline{c_n} e^{im\theta} e^{-in\theta}$$

$$= |f(\theta)|^2 - \sum_{-N}^{N} \left[f(\theta)\overline{c_n} e^{-in\theta} + \overline{f(\theta)} c_n e^{in\theta} \right]$$

$$+ \sum_{n,m=-N}^{N} c_m \overline{c_n} e^{i(m-n)\theta}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} \, d\theta$$

Integrate

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} c_m \bar{c}_n e^{i(m-n)\theta}\, d\theta = \begin{cases} 0 & \text{if } n \neq m \\ c_m \bar{c}_n = |c_n|^2 & \text{if } n = m \end{cases}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, \bar{c}_n e^{-in\theta}\, d\theta = \frac{1}{2\pi}\bar{c}_n \int_{\pi}^{\pi} f(\theta) e^{-in\theta} = \bar{c}_n c_n = |c_n|^2$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(\theta)}\, c_n e^{in\theta}\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, \bar{c}_n e^{-in\theta} = |c_n|^2$$

Hence

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left| f(\theta) - \sum_{-N}^{N} c_n e^{in\theta} \right|^2 d\theta = ?$$

Hence

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f(\theta)-\sum_{-N}^{N}c_n e^{in\theta}\right|^2 d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta)|^2 d\theta$$

$$-\sum_{-N}^{N}2|c_n|^2+\sum_{-N}^{N}|c_n|^2$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta)|^2 d\theta-\sum_{-N}^{N}|c_n|^2 \geqslant 0$$

Thus for all N

$$\sum_{-N}^{N}|c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta)|^2 d\theta$$

letting $N \to \infty$ we conclude Bessel's inequality.

$$\lim_{n \to \pm\infty}|c_n|^2 = 0.$$