Separation of variables

Solve the heat equation using separation of variables. Consider $u(x,t) = X(x)T(t)$ satisfying $u_t = k u_{xx}$ for x in $[0,l]$

then $$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = A \Rightarrow \begin{cases} T'(t) = AkT(t) \\ X''(x) = AX(x) \end{cases}$$

$A<0$ $\quad \lambda = \sqrt{-A}$ $$\begin{cases} T(t) = C_0 e^{Akt} \\ X(x) = C_1 \cos \lambda x + C_2 \sin \lambda x \end{cases}$$

$A>0$ $\quad \mu = \sqrt{A}$ $$\begin{cases} T(t) = C_0 e^{Akt} \\ X(x) = C_1 e^{\mu x} + C_2 e^{-\mu x} \end{cases}$$

Question: Are there some of these solutions that satisfy given boundary conditions?

$A<0 \quad \lambda = \sqrt{-A} \quad u(x,t) = e^{Akt}\left(C_1 \cos \lambda x + C_2 \sin \lambda x\right)$

(1) $u(0,t) = X(0)T(t) = u(l,t) = X(l)T(t) = 0$ for all t

$\Rightarrow X(0) = X(l) = 0$, for each $n = 1, 2, \ldots$

$A = -\frac{n^2\pi^2}{l^2}$ produces a solution of the form

$$u_n(x,t) = \exp\left(\frac{-n^2\pi^2 k t}{l^2}\right) \sin\frac{n\pi x}{l}$$

building blocks

(2) $u_x(0,t) = X'(0)T(t) = u_x(l,t) = X'(l)T(t) = 0$

$\Rightarrow X'(0) = X'(l) = 0$, for each $n = 1, 2, \ldots$

$A = -\frac{n^2\pi^2}{l^2}$ produces a solution of the form

$$u_n(x,t) = \exp\left(\frac{-n^2\pi^2 k t}{l^2}\right) \cos\frac{n\pi x}{l}$$

building blocks

$A > 0 \qquad \mu = \sqrt{A} \qquad u(x,t) = e^{Akt}\left(C_1 e^{\mu x} + C_2 e^{-\mu x}\right)$

(1) $u(0,t) = X(0)T(t) = u(l,t) = X(l)T(t) = 0$ for all t

$\Rightarrow X(0) = X(l) = 0$

$\Rightarrow C_1 + C_2 = 0 \quad \& \quad C_1 e^{l\mu} + C_2 e^{-l\mu} = 0$

$\Rightarrow C_2 = -C_1 \quad \& \quad C_1\left(e^{l\mu} - e^{-l\mu}\right) = 0$

since $l \neq 0$ & $\mu \neq 0$

$\Rightarrow C_1 = 0 \Rightarrow C_2 = 0$

$u(x,t) \equiv 0$

no non-trivial solutions of this form.

(2) Same result with $u_x(0,t) = u_x(l,t) = 0$.

Q: Could we have predicted this outcome?

Summarizing : formally

$$u(x,t) = \sum_{n=1}^{\infty} a_n \exp\left(\frac{-n^2\pi^2 k t}{l}\right) \sin\frac{n\pi x}{l}$$ is a "solution"

of $u_t = k u_{xx}$ with boundary conditions $u(0,t) = u(l,t) = 0$

The initial condition plays a key role to determine the solution

Assume

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} a_n \sin\frac{n\pi x}{l}$$

key point of this course; how given f do we find a_n? does the series converge? in what sense?

Is there something special to sine and cosine?

1- Suppose u_1, u_2 are solutions of the linear differential equation $Lu = f$ where $f \neq 0$ - When is $c_1u_1 + c_2u_2$ also a solution?

$$L(c_1u_1 + c_2u_2) = c_1Lu_1 + c_2Lu_2 = c_1f + c_2f$$
$$= (c_1 + c_2)f = f$$

$$\Longrightarrow \boxed{c_1 + c_2 = 1}$$

2. Consider the nonlinear differential equation $u' = u(1-u)$

a- Show that $u_1(x) = \frac{e^x}{1+e^x}$ & $u_2(x) = 1$ are solutions

b- Show $u_1 + u_2$ is not a solution

c- When is cu_1 a solution? How about cu_2?

a) u_2 ✓ $\quad u_1'(x) = \frac{e^x}{1+e^x} + \frac{e^x(-e^x)}{(1+e^x)^2} = \underbrace{\frac{e^x}{1+e^x}}_{u_1}\left(1 - \underbrace{\frac{e^x}{1+e^x}}_{u_1}\right)$

b) $(u_1+u_2)' = u_1' + u_2' = u_1' = u_1(1-u_1)$

$(u_1+u_2)(1-u_1-u_2) = -u_1(u_1+u_2)$ #

c) for what c do we have $cu' = cu(1-cu)$ if $u' = u(1-u)$

& $u \neq 0$

$cu(1-u) = cu(1-cu)$ / $\quad c=0$

$c \neq 0 \qquad (1-u) = 1-cu \Rightarrow \boxed{c=1}$

3. Find linear differential operators L & M for which

$L(M(u)) \neq M(L(u))$ for all $u \neq 0$

$Lu = xu'$ $\qquad M(u) = x^2 u$

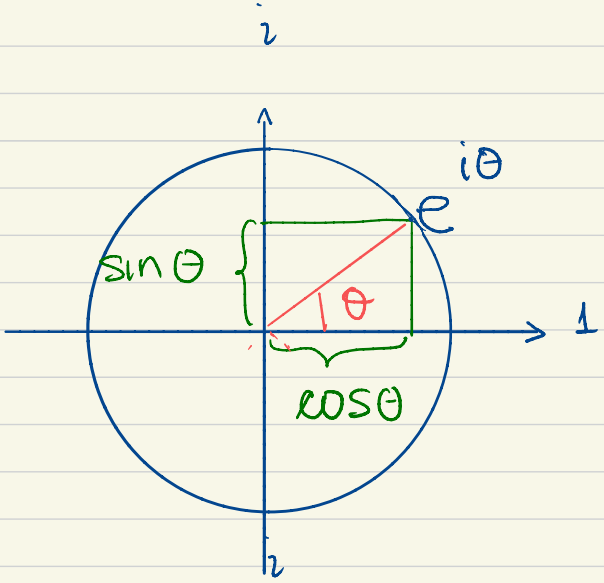
$M(Lu) = M(xu') = x^3 u'$

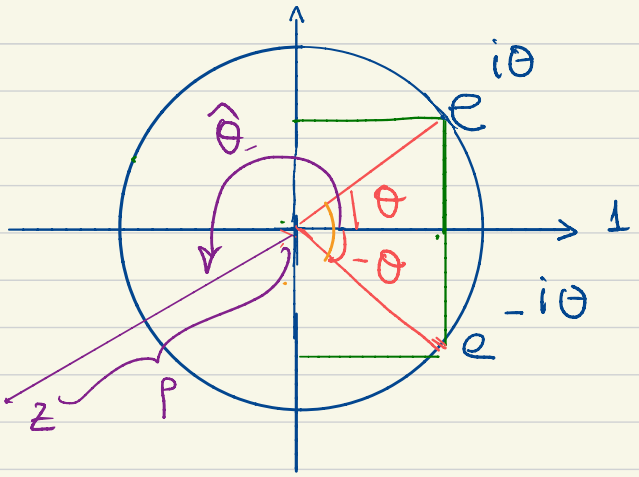
$L(Mu) = L(x^2 u) = x(x^2 u)' = x(x^2 u' + 2xu)$

$= x^3 u' + 2x^2 u$

$M(Lu) - L(Mu) = -2x^2 u \neq 0$

Fourier series of a periodic function





$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\left| e^{i\theta} \right|^2 = e^{i\theta} e^{-i\theta} = 1$$

$$e^{i(\theta + 2\pi)} = e^{i\theta}$$

$$z = \rho e^{i\hat{\theta}}$$

$$|z|^2 = z\bar{z} = \rho e^{i\hat{\theta}} \cdot \rho e^{-i\hat{\theta}} = \rho^2$$

$f : \mathbb{R} \to \mathbb{C}$ Riemann integrable on bounded intervals (for example f bounded and continuous except at most in finitely many points in each interval).

Assume f is periodic with period 2π. $(f(\theta + 2\pi) = f(\theta))$

GOAL: expand f as a series of the form

$$f(\theta) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

Note that $\frac{1}{2} a_0 = \frac{1}{2} a_0 \cos(0\theta) + b_0 \sin(0\theta)$

convenient later

Why do we want to do this?

$f : \mathbb{R} \to \mathbb{C}$ Riemann integrable on bounded intervals (for example f bounded and continuous except at most in finitely many points in each interval).

Assume f is periodic with period 2π.

GOAL: expand f as a series of the form

$$f(\theta) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

Why? It is a decomposition of a 2π-periodic function f into "simple" periodic 2π-periodic functions $\cos n\theta$ and $\sin n\theta$, $n = 1, 2 \ldots$

building blocks.

If

(1) $$f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

since

$$\cos n\theta = \frac{e^{in\theta} + e^{-in\theta}}{2} \quad \& \quad \sin n\theta = \frac{e^{in\theta} - e^{-in\theta}}{2i}$$

then

(2) $$\boxed{f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}}$$ where for $n = 1, 2, \ldots$

(1) $\Rightarrow$ (2)

$$c_0 = \frac{1}{2}a_0 \quad , \quad c_n = \frac{1}{2}(a_n - ib_n) \quad , \quad c_{-n} = \frac{1}{2}(a_n + ib_n)$$

(2) $\Rightarrow$ (1) using $\cos(-n\theta) = \cos n\theta$ (even) , $\sin(-n\theta) = -\sin n\theta$ (odd)

$$a_0 = 2c_0 \quad , \quad a_n = c_n + c_{-n} \quad , \quad b_n = i(c_n - c_{-n})$$

Linear algebra recap:

- linear maps
- basis: building blocks
- orthonormal basis
- inner product

For $n, k = \pm 1, \pm 2, \ldots$ consider

$$\int_{-\pi}^{\pi} e^{i(n-k)\theta}\, d\theta = \frac{1}{i(n-k)}\, e^{i(n-k)\theta}\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{i(n-k)}\left[e^{i(n-k)\pi} - e^{-i(n-k)\pi} \right]$$

$$e^{i(n-k)\pi} = \underbrace{\cos(n-k)\pi}_{(-1)^{n-k}} + \underbrace{i \sin(n-k)\pi}_{0}$$

$$\left\{ e^{in\theta} \right\}_{n=\pm 1, \pm 2, \ldots} \qquad \langle e^{in\theta}, e^{ik\theta} \rangle = \int_{-\pi}^{\pi} e^{in\theta} e^{-ik\theta}\, d\theta$$

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\, \bar{g}(x)\, dx$$

$$\int_{-\pi}^{\pi} e^{i(n-k)\theta}\, d\theta = \begin{cases} \dfrac{(-1)^{n-k} - (-1)^{-(n-k)}}{i(n-k)} = 0 & \text{if } n \neq k \\ 2\pi & \text{if } n = k \end{cases}$$

Two questions : ① If f is 2π-periodic, does f admit a series expansion of the form

(★)
$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

↳ what does this equality really mean ?

② If f admits such a series expansion how can one compute c_n ?

Formal computation: if (⋆) holds and we can integrate the series term we have for $k = \pm 1, \pm 2, \ldots$

$$\int_{-\pi}^{\pi} f(\theta)\, e^{-ik\theta}\, d\theta = \int_{-\pi}^{\pi} \left(\sum_{n=-\infty}^{\infty} c_n e^{in\theta} e^{-ik\theta} \right) d\theta$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} c_n e^{i\theta(n-k)}\, d\theta = \sum_{n=-\infty}^{\infty} c_n \underbrace{\int_{-\pi}^{\pi} e^{i(n-k)}\, d\theta}$$

$$= c_k\, 2\pi$$

$$2\pi c_k = \int_{-\pi}^{\pi} f(\theta)\, e^{-ik\theta}\, d\theta$$

Formal computation: if (✱) holds and we can integrate the series term we have for $k = \pm 1, \pm 2, \ldots$

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, e^{-ik\theta}\, d\theta$$

Find $a_0 = 2c_0$, $a_n = c_n + c_{-n}$, $b_n = i(c_n - c_{-n})$

$$a_n = \frac{1}{2\pi} \underbrace{\left(\int_{-\pi}^{\pi} f(\theta)\, e^{in\theta}\, d\theta + \int_{-\pi}^{\pi} f(\theta)\, e^{-in\theta}\, d\theta \right)}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \underbrace{\left(e^{in\theta} + e^{-in\theta} \right)}_{2\cos n\theta} d\theta$$

Formal computation: if (✱) holds and we can integrate the series term we have for $n = \pm 1, \pm 2, \ldots$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, e^{-in\theta}\, d\theta$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta \qquad n \geq 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta \qquad n \geq 1$$