Some equations of mathematical physics

Wave equation

$$u_{tt} = \frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$

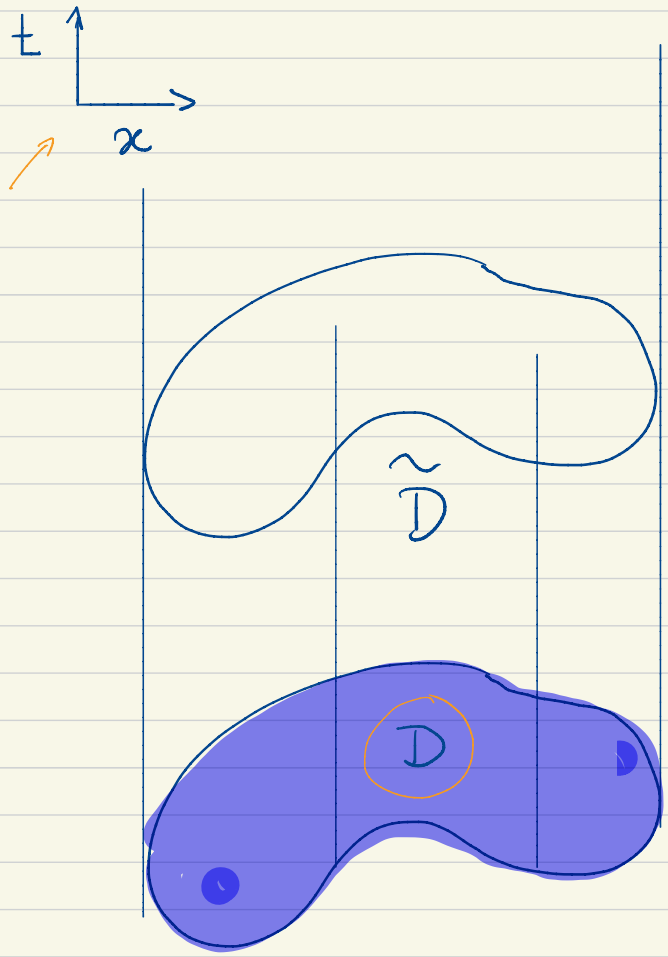
Heat equation

$$u_t = k \Delta u$$

Laplace equation

$$\Delta u = 0$$

Boundary conditions



Solve the heat equation $u_t = \Delta u$ in $\tilde{D}$ subject to the initial condition

$$u(x,0) = f(x)$$

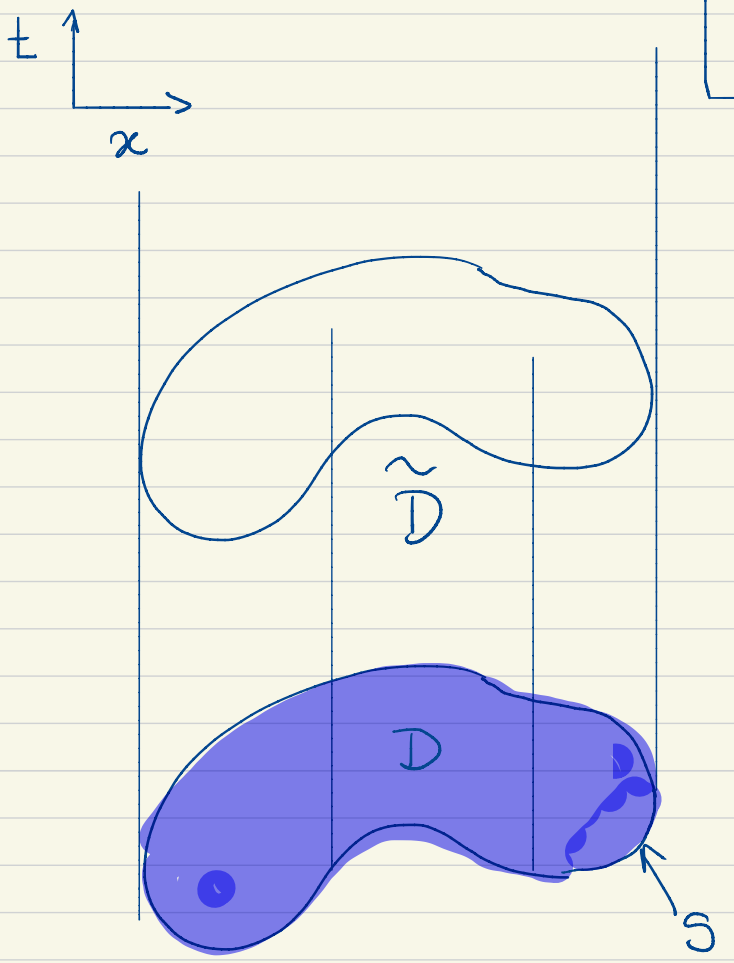
↑ temperature distribution at time $t=0$

← $u(x,0) = f(x)$ condition on the horizontal portion of the boundary of $\tilde{D}$

D region in x-space

$\tilde{D} = \{(x,t) : x \in D, t > 0\}$

Boundary conditions (Heat equation)



D region in x-space

$\tilde{D} = \{(x,t) : x \in D, t > 0\}$

(x is an element of D)

Solve the heat equation $u_t = \Delta u$ in $\tilde{D}$ subject to the initial condition

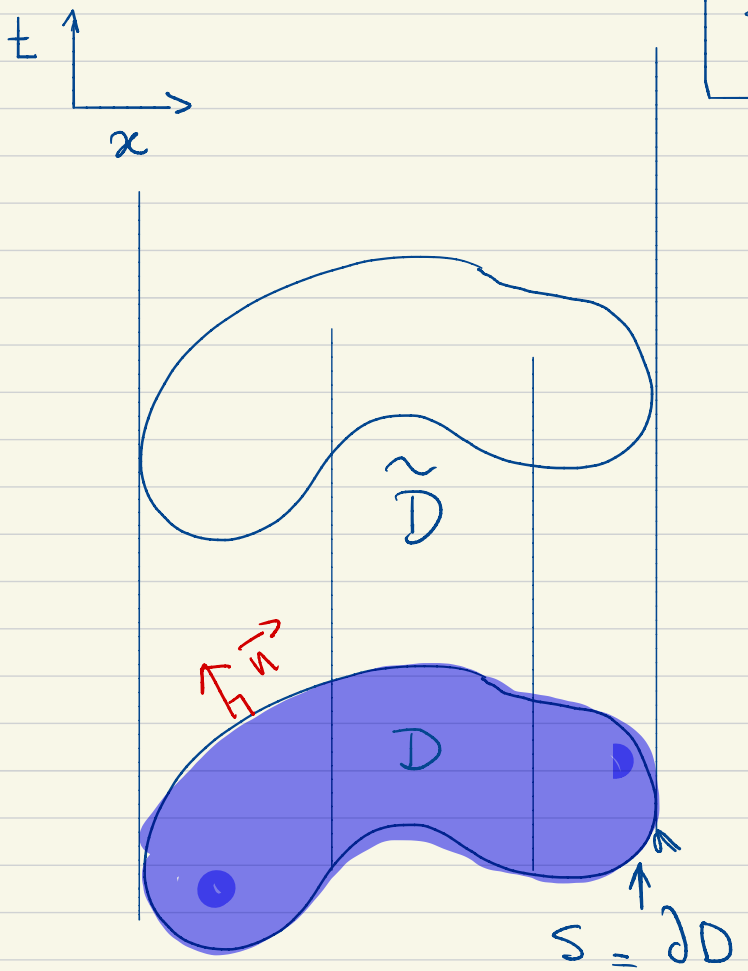
$u(x,0) = f(x)$

temperature distribution at time $t=0$

Condition on the "vertical" part of the boundary of $\tilde{D}$ describes what happens to the heat when it reaches the boundary surface S of the spacial region D. We assume it is constant

$u(x,t) = u_0$ for $x \in S$ and $t > 0$.

Boundary conditions (Heat equation)



$$\begin{cases} u_t = \Delta u & \text{in } \tilde{D} \\ u(x,0) = f(x) & \text{for } x \in D \end{cases}$$

If D is insulated : no heat can flow in or out

$$(\nabla u \cdot \vec{n})(x,t) = 0 \quad \text{for } x \in \partial D \quad t > 0$$

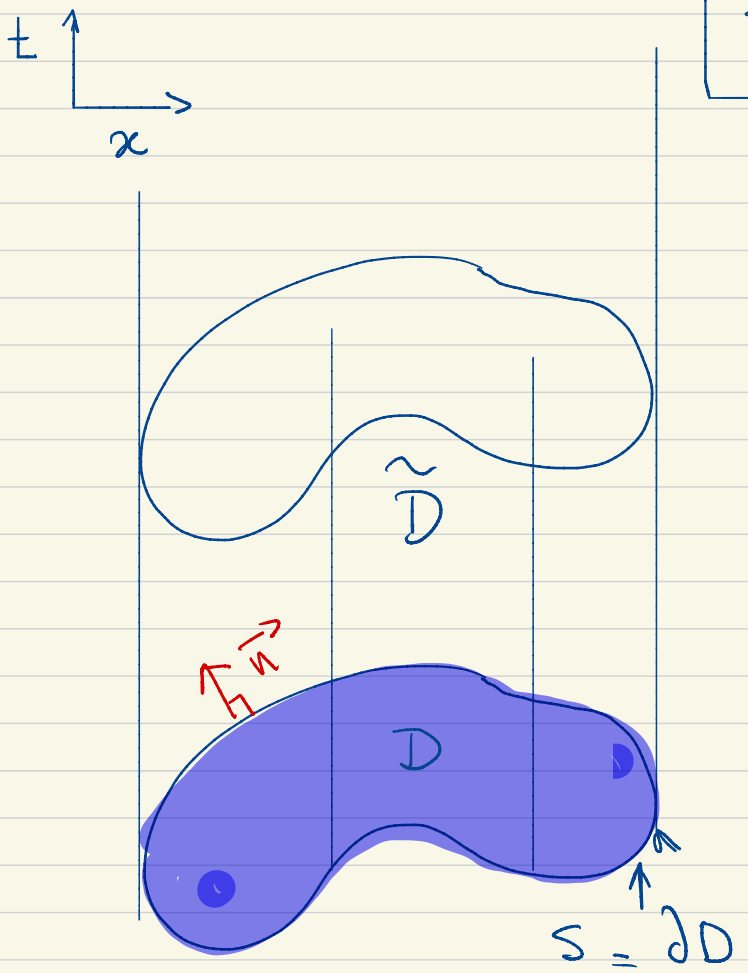
normal derivative of u

(inner product of the gradient of u and the outward unit normal to ∂D)

D region in x-space

$\tilde{D} = \{(x,t) : x \in D, t > 0\}$

Boundary conditions (Heat equation)



D region in x-space

$\tilde{D} = \{(x,t) : x \in D, t > 0\}$

$$u_t = \Delta u \quad \text{in} \quad \tilde{D}$$

$$u(x,0) = f(x) \quad \text{for} \quad x \in D$$

A more realistic assumption is that the region outside D is held at constant temperature u_0, and that the rate of heat flow across ∂D is proportional to the difference of the temperature on both sides

$$(\nabla u \cdot \vec{n})(x,t) + a(u(x,t) - u_0) = 0$$

for $x \in \partial D$ and $t > 0$

Newton's law of cooling

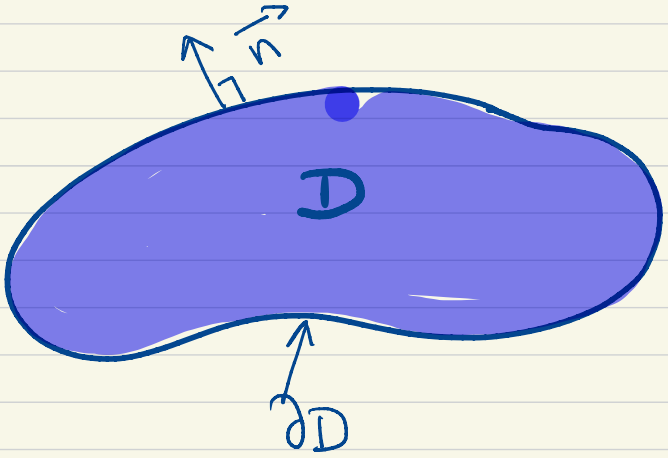
$$\begin{cases} u_t = \Delta u & \text{in } \widetilde{D} \\ u(x,0) = f(x) & \text{for } x \in D \\ u(x,t) = u_0 & \text{for } x \in \partial D,\ t > 0 \end{cases}$$

$$\begin{cases} u_t = \Delta u & \text{in } \widetilde{D} \\ u(x,0) = f(x) & \text{for } x \in D \\ \nabla u \cdot \vec{n}\,(x,t) = 0 & \text{for } x \in \partial D,\ t > 0 \end{cases}$$

$$\begin{cases} u_t = \Delta u & \text{in } \widetilde{D} \\ u(x,0) = f(x) & \text{for } x \in D \\ (\nabla u \cdot \vec{n})(x,t) + a\,(u(x,t) - u_0) = 0 & \text{for } x \in \partial D\ t > 0 \end{cases}$$

are well posed problems: that is problems having a unique solution depending continuously on the initial data f.

Boundary conditions (laplace equation)



Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = f & \text{on } \partial D \end{cases}$$

Neumann problem

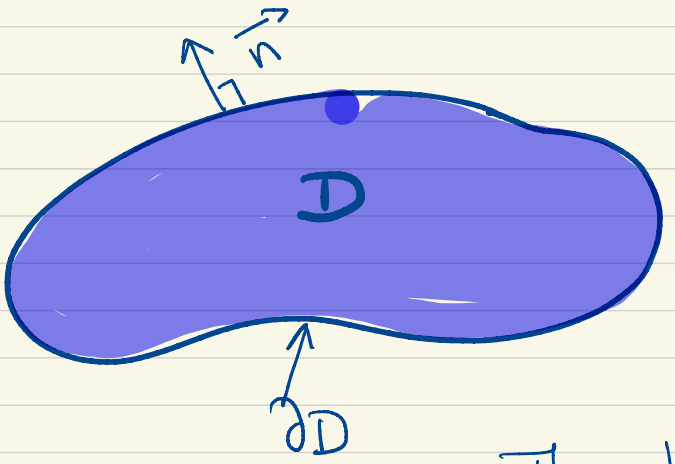
$$\begin{cases} \Delta u = 0 & \text{in } D \\ \nabla u \cdot \vec{n} = g & \text{on } \partial D \end{cases}$$

Note that if u is a solution to the Neumann problem so is $u + C$, where C is constant.

$$\Delta(u+C) = \Delta u + \Delta C = \Delta u + 0 = \Delta u$$

$$\nabla(u+C) \cdot \vec{n} = \nabla u \cdot \vec{n} = g \qquad \text{bec } \nabla C = 0$$

Boundary conditions (laplace equation)



Neumann problem

$$\begin{cases} \Delta u = 0 & \text{in } D \\ \nabla u \cdot \vec{n} = g & \text{on } \partial D \end{cases}$$

The boundary data g must satisfy $\int_{\partial D} g = 0$

in fact recall the divergence theorem

$$\int_{\partial D} g = \int_{\partial D} \nabla u \cdot \vec{n} \, ds = \int_D \operatorname{div} \nabla u \, dv = \int_D \Delta u \, dv = 0$$

$$\operatorname{div} \nabla u = \operatorname{div} (u_{x_1} \dots u_{x_n})$$

$$= u_{x_1 x_1} + \dots + u_{x_n x_n} = \Delta u$$

The wave, heat and Laplace equations can be generalized by adding a right hand side; i.e

$$u_{tt} - c^2 \Delta u = F(x,t)$$

$$u_t - c \Delta u = F(x,t)$$

$$\Delta u = F(x,t)$$

Inhomogeneous wave, heat and Laplace equations. These equations can be written as

$$Lu = F$$

, L is above is a second order partial differential operator.

Linear differential operators

A second order linear partial differential operator is an operation that transforms a function u of the variables $\mathbf{x} = (x_1, \ldots, x_n)$ into another function Lu given by

$$Lu = a(x)u + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}$$

partial differential

second order

Linear (same as linear transformation in linear algebra) means that if $u_1, \ldots, u_k$ are admissible functions and $c_1, \ldots c_k$ are constants then

$$L(c_1 u_1 + \cdots + c_k u_k) = c_1 L u_1 + \cdots + c_k L u_k$$

$$Lu = x^2 u + x^3 u''$$

$$L(u+v) = x^2 (u+v) + x^3 \underbrace{(u+v)''}_{u'' + v''}$$

$$= x^2 u + x^2 v + x^3 u'' + x^3 v''$$

$$= Lu + Lv$$

A (second order) linear partial differential equation is an equation of the form

$$\boxed{Lu = F}$$ where F is a function of x.

- The equation is homogeneous is $F = 0$
- The equation is nhomogeneous otherwise

The boundary conditions are of the form $B(u) = f$ where B is a linear differential operator

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Superposition Principle: If $u_1, \dots, u_k$ satisfy for $j = 1, \dots, k$

$Lu_j = F_j$ with boundary condition $B(u_j) = f_j$ and $c_1, \dots, c_k$ are constants then $u = c_1 u_1 + \dots + c_k u_k$ satisfies

$$Lu = c_1 F_1 + \dots + c_k F_k \quad \text{and} \quad B(u) = c_1 f_1 + \dots + c_k f_k$$

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$$Lu = c_1 F_1 + \dots + c_k F_k \quad \text{and} \quad B(u) = c_1 f_1 + \dots + c_k f_k$$

$$\begin{cases} Lu = F & \text{in } D \\ Bu = f & \text{on } \partial D \end{cases}$$

how does this help?

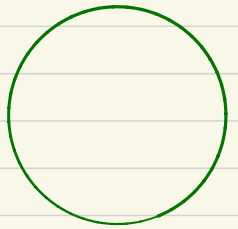
$$\begin{cases} Lu_0 = 0 & \text{in } D \\ Bu_0 = 0 & \text{on } \partial D \end{cases}$$

$L = \Delta$ $\begin{cases} \Delta u_0 = 0 & \text{on } B_1 \\ u_0 = 0 & \text{on } \partial B_1 \end{cases}$

guess a specific one

$$\begin{cases} Lv = F & \text{in } D \\ Bv = f & \text{on } \partial D \end{cases}$$

$u_0 + v$ will be a solution



$u_0 \equiv 0$

Superposition Principle: If $u_1, \dots, u_k$ satisfy for $j = 1, \dots, k$ $Lu_j = F_j$ with boundary condition $B(u_j) = f_j$ and $c_1, \dots, c_k$ are constants then $u = c_1 u_1 + \dots + c_k u_k$ satisfies

$$Lu = c_1 F_1 + \dots + c_k F_k \quad \text{and} \quad B(u) = c_1 f_1 + \dots + c_k f_k$$

$$\begin{cases} Lu = F & \text{in } D \\ Bu = f & \text{on } \partial D \end{cases}$$

how does this help?

$$\begin{cases} Lu_1 = F \\ Bu_1 = 0 \end{cases} \qquad \begin{cases} Lu_2 = 0 \\ Bu_2 = f \end{cases}$$

$$L(u_1 + u_2) = Lu_1 + Lu_2 = F + 0 = F$$

$$B(u_1 + u_2) = Bu_1 + Bu_2 = 0 + f = f$$

Application of the Superposition principle

To solve $Lu = F$, $B(u) = f$ (*)

(1) Find solutions to $Lu_0 = 0$, $B(u_0) = 0$ then find one particular solution v to $Lv = F$, $B(v) = f$ then for each u_0, $u = u_0 + v$ is a solution of (*) and all solutions are of this form.

If $Lu_0 = 0$, $B(u_0) = 0$ & $Lv = F$, $B(v) = f$

then if $u = u_0 + v$, $Lu = Lu_0 + Lv = 0 + F = F$

$B(u) = Bu_0 + Bv = 0 + f = f$

If w is a solution of (*), let $u_0 = w - v$ with v as above then

$Lu_0 = Lw - Lv = F - F = 0$, $B(u_0) = B(w) - B(v) = f - f = 0$

$w = u_0 + v$

Application of the Superposition principle

To solve $Lu = F$, $B(u) = f$ (*)

(1) Find solutions to $Lu_0 = 0$, $B(u_0) = 0$ then find one particular solution v to $Lv = F$, $B(v) = f$ then for each u_0, $u = u_0 + v$ is a solution of (*) and all solutions are of this form.

(2) Find solutions $Lu_1 = F$, $B(u_1) = 0$

$Lu_2 = 0$, $B(u_2) = f$

then $u = u_1 + u_2$ solves (*)

In general methods in this class well adapted to find u_0, u_1 or u_2.

Separation of variables

Suppose we are looking for solutions of $Lu = 0$ & $B(u) = 0$ in 2-space - Goal find solutions of the form

$$u(x, y) = X(x)\, Y(y)$$

The method works if when we substitute this formula for u, we find 2 ODEs that we can solve -

If there are 3 independent variables x, y, z we look for solutions of the form

$$u(x, y, z) = X(x)\, v(y, z)$$

then look for solutions of the equation for v of the form

$$v(y, z) = Y(y)\, Z(z)$$

Solutions to some ODEs

(1) $f' = af \implies f(x) = Ce^{ax}$

(2) Find the general solution of $f'' + af' + bf = 0$.

look for solutions of the form $f(x) = e^{rx}$

$f'(x) = re^{rx}$ $\quad f''(x) = r^2 e^{rx}$ - Plug in

$$f''(x) + af'(x) + bf(x) = r^2 e^{rx} + are^{rx} + be^{rx}$$
$$= (r^2 + ar + b)e^{rx} = 0$$

$\implies r^2 + ar + b = 0$ then if $a^2 - 4b \neq 0$

$$r_1 = \frac{-a + \sqrt{a^2 - 4b}}{2} \neq r_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}$$

2 distinct sol.

The general solution to $f'' + af' + bf = 0$ if $a^2 - 4b \neq 0$ is $f(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$ (as $e^{r_1 x}, e^{r_2 x}$ are linearly independent solutions)

If $a^2 - 4b = 0$ then $r_1 = r_2 = -\frac{a}{2}$ then $xe^{-a/2 x}$ is also a solution - In this case

$$f'' + af' + bf = f'' + af' + \frac{a^2}{4} f \quad \text{and if}$$

$$f(x) = xe^{-a/2 x} \qquad f'(x) = e^{-a/2 x} - a/2\, xe^{-a/2 x}$$

$$f''(x) = -a/2\, e^{-a/2 x} - \frac{a}{2} e^{-a/2 x} + \frac{a^2}{4} xe^{-a/2 x}$$

$$f''(x) = -ae^{-a/2 x} + a^2/4\, xe^{-a/2 x}$$

$$f'' + af' + \frac{a^2}{4} f = \underbrace{-ae^{-a/2 x}} + \frac{a^2}{4} xe^{-a/x} + \underbrace{ae^{-a/2 x}} - \frac{a^2}{2} xe^{-a/2 x}$$

$$+ a^2/4\, xe^{-a/2 x}$$

Theorem: The general solution of $f'' + af' + bf = 0$ is $f(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$ where r_1, r_2 are the roots of the equation $r^2 + ar + b = 0$ and C_1, C_2 are arbitrary complex numbers. If $r_1 = r_2$ the general solution is $(C_1 + C_2 x) e^{r_1 x}$.

Note r_1 and r_2 can be complex, $a^2 - 4b < 0$

$$r_1 = -\frac{a}{2} + \frac{\sqrt{a^2-4b}}{2} \quad \& \quad r_2 = -\frac{a}{2} - \frac{\sqrt{a^2-4b}}{2}$$

$$r_1 = \rho + i\sigma \quad \& \quad r_2 = \rho - i\sigma \qquad \rho = -a/2, \; \sigma = \frac{\sqrt{4b-a^2}}{2}$$

$$e^{r_1 x} = e^{\rho x} e^{i\sigma x} = e^{\rho x}(\cos \sigma x + i \sin \sigma x)$$

$$e^{r_2 x} = e^{\rho x} e^{-i\sigma x} = e^{\rho x}(\cos \sigma x - i \sin \sigma x)$$

Theorem : The general solution of $f'' + af' + bf = 0$ is $f(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$ where r_1, r_2 are the roots of the equation $r^2 + ar + b = 0$ and C_1, C_2 are arbitrary complex numbers. If $r_1 = r_2$ the general solution is $(C_1 + C_2 x) e^{r_1 x}$.

Remarks

(i) If $r_1 = \rho + i\sigma$ and $r_2 = \rho - i\sigma$ the general solution is

$$e^{\rho x} (\tilde{C}_1 \cos \sigma x + i \tilde{C}_2 \sin \sigma x)$$

(ii) If $\alpha > 0$ the general solution of

- $f'' + \alpha^2 f = 0$ is $C_1 \cos \alpha x + C_2 \sin \alpha x$
- $f'' - \alpha^2 f = 0$ is $C_1 \cosh \alpha x + C_2 \sinh \alpha x$

For $f'' + \alpha^2 f = 0$ then $r_1 = \frac{\sqrt{-4\alpha^2}}{2} = i\alpha$ and

$r_2 = -i\alpha$ $(\rho = 0, \sigma = \alpha)$

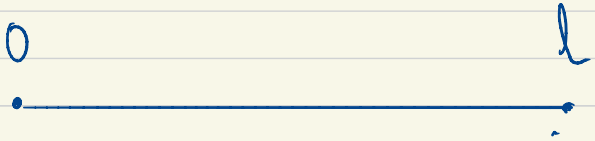
For $f'' - \alpha^2 f = 0$ then $r_1 = \frac{\sqrt{4\alpha^2}}{2} = \alpha$ & $r_2 = -\alpha$

then

$$C_1 e^{\alpha x} + C_2 e^{-\alpha x} = (C_1 + C_2)\frac{e^{\alpha x} + e^{-\alpha x}}{2} + (C_1 - C_2)\frac{e^{\alpha x} - e^{-\alpha x}}{2}$$

$$= \tilde{C}_1 \cosh \alpha x + \tilde{C}_2 \sinh \alpha x$$

One dimensional heat flow - 2 different types of boundary conditions



= rod of length l

(1) $\begin{cases} u_t = k u_{xx} \\ u(0,t) = u(l,t) = 0 \end{cases}$ vs (2) $\begin{cases} u_t = k u_{xx} \\ u_x(0,t) = u_x(l,t) = 0 \end{cases}$

↑ ends of the rod held at temperature 0

↑ ends of the rod are insulated

In both cases we are looking for solutions of the form

$u(x,t) = X(x)\,T(t)$ with X & T not identically 0

satisfying $u_t = k u_{xx}$

if $u(x,t) = X(x)T(t)$ and $u_t = k u_{xx}$ then

$u_t = X(x)T'(t)$ & $= X''(x)T(t)$

thus $\boxed{X(x)T'(t) = k X''(x)T(t)}$

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = A$$

↑ only depends on x

↑ only depends on t

$\Longrightarrow$ both sides must be constant

if $u(x,t) = X(x)T(t)$ and $u_t = k u_{xx}$ then

$u_t = X(x)T'(t)$ & $u_{xx} = X''(x)T(t)$

thus $\boxed{X(x)T'(t) = k X''(x)T(t)}$

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = A \implies \begin{cases} T'(t) = Ak\,T(t) \\ X''(x) = A\,X(x) \end{cases}$$

$A<0$

$$\implies \begin{cases} T(t) = C_0 e^{Akt} \\ X(x) = C_1 \cos \lambda x + C_2 \sin \lambda x \qquad \lambda = \sqrt{-A} \end{cases}$$

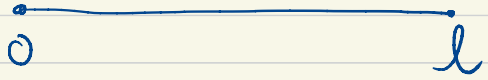
$A>0$

$$\implies \begin{cases} T(t) = C_0 e^{Akt} \\ X(x) = C_1 e^{\mu x} + C_2 e^{-\mu x} \qquad \mu = \sqrt{A} \end{cases}$$

$u(x,t) = C_0 e^{Akt}(C_1 \cos \lambda x + C_2 \sin \lambda x)$ $\qquad \lambda = \sqrt{-A}$

$C_0 \neq 0$

$u_t = k u_{xx}$



$$u(0,t) = u(l,t) = 0$$

$$u(0,t) = C_0 e^{Akt}(C_1 \underbrace{\cos \lambda 0}_{1} + C_2 \cancel{\sin \lambda 0}^{0}) = C_0 e^{Akt} C_1 = 0$$

$$\boxed{C_1 = 0}$$

$$u(x,t) = C_0 e^{Akt} C_2 \sin \lambda x$$

$$u(l,t) = C_0 e^{Akt} C_2 \sin \lambda l = 0$$

\# \# \#

0 0 0

$$\boxed{\lambda l = n\pi}$$

integer

if $A < 0$ $\quad X(x) = C_1 \cos \lambda x + C_2 \sin \lambda x \quad \lambda = \sqrt{-A}$

(1) $X(0)T(t) = X(l)T(t) = 0$

$\Rightarrow X(0) = X(l) = 0$

(2) $X'(0)T(t) = X'(l)T(t) = 0$

$\Rightarrow X'(0) = X'(l) = 0$

$X(0) = C_1 = 0$

$X(l) = C_2 \sin \lambda l = 0$

$\sqrt{-A}\, l = \lambda l = n\pi \qquad n \in \mathbb{Z}$

$$A = -\frac{n^2\pi^2}{l^2}$$

$$\boxed{u_n(x,t) = \exp\left(-\frac{n^2\pi^2 kt}{l^2}\right) \sin\frac{n\pi x}{l}}$$

$n = 1, 2 \ldots$

↑ building blocks

$X'(x) = -\lambda C_1 \sin \lambda x + C_2 \lambda \cos \lambda x$

$X'(0) = 0 \Rightarrow C_2 = 0$

$X'(l) = 0 \Rightarrow \sin \lambda l = 0$

continue and find the building blocks