Some equations of mathematical physics

The wave equation provides a mathematical model for:

(i) Vibrations of a streched string / stretched membrane
(ii) Vibrations of an incompressible fluid (water)
(iii) Sound waves
(iv) Electromagnetic Waves (light or radio waves)

Key ingredient - the **Laplacian**: if u is a function of real variables $x_1, \dots x_n$ of class C^2 (that is 2 continuous derivatives) the **Laplacian** of u is the function

$$\nabla^2 u = \Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \dots + \frac{\partial^2 u}{\partial x_n^2}$$

physics notation

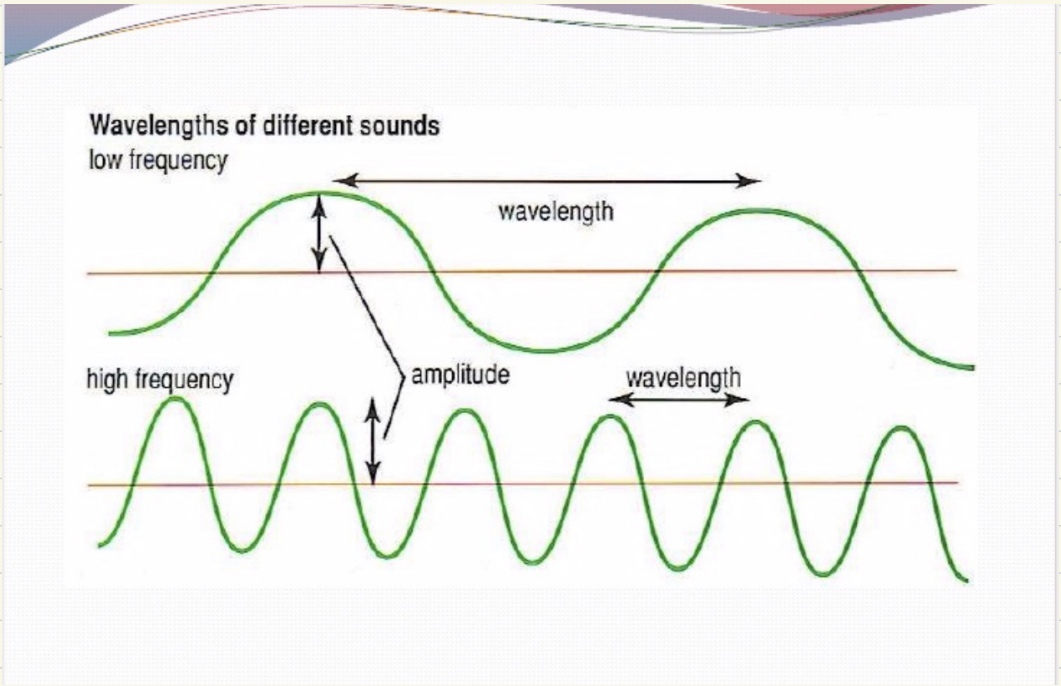
math notation

Wave equation

$$u_{tt} = \frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$

t = time

The derivation of this equation in the examples above involves making some simplifying assumptions.



Heat equation

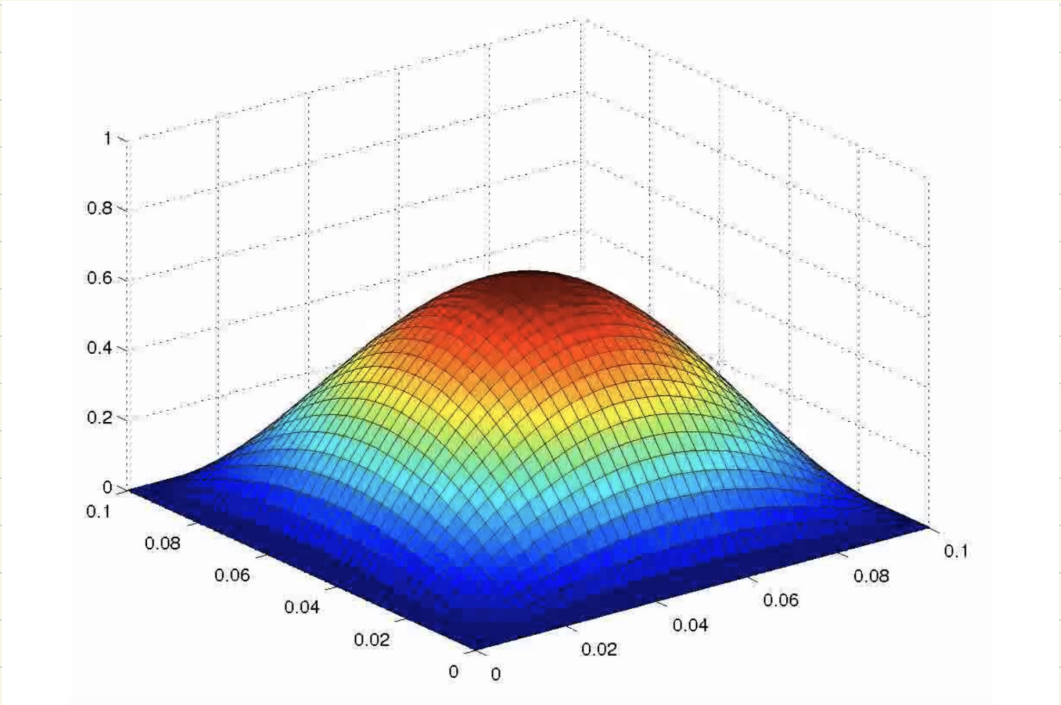
$$u_t = k \Delta u$$

This describes the diffusion of thermal energy in a homogeneous material.

$u(x,t)$ = temperature at a position x and time t.

k = thermal diffusivity of the material





Laplace equation

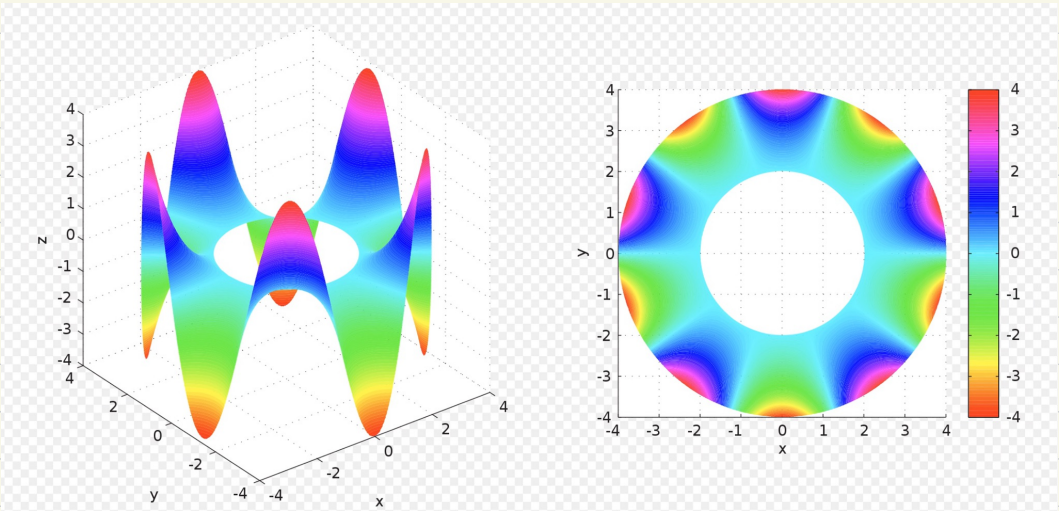
$$\Delta u = 0$$

It is satisfied by:

(i) electrostatic potential in any region containing no electric charge

(ii) standing waves

(iii) steady-state heat distribution



Example 1: $u(x,y,t) = t^{-1} e^{-\frac{1}{4kt}(x^2+y^2)}$ satisfies the heat equation $u_t = k(u_{xx} + u_{yy})$ for $t > 0$

$$u_t = -\frac{1}{t^2} e^{-\frac{1}{4kt}(x^2+y^2)} + \frac{1}{t}\left(\frac{x^2+y^2}{4kt^2}\right) e^{-\frac{1}{4kt}(x^2+y^2)}$$

$$u_t = e^{-\frac{1}{4kt}(x^2+y^2)} \left(\frac{x^2+y^2}{4kt^3} - \frac{1}{t^2}\right)$$

$$u_x = -\frac{2x}{4kt^2} e^{-\frac{1}{4kt}(x^2+y^2)}$$

$$u_{xx} = -\frac{2}{4kt^2} e^{-\frac{1}{4kt}(x^2+y^2)} + \frac{4x^2}{(4k)^2 t^3} e^{-\frac{1}{4kt}(x^2+y^2)}$$

$$u_{xx} + u_{yy} = e^{-\frac{1}{4kt}(x^2+y^2)} \left(-\frac{1}{kt^2} + \frac{x^2+y^2}{4k^2t^3}\right)$$

$$= \frac{1}{k}\left(\frac{x^2+y^2}{4kt^3} - \frac{1}{t^2}\right) e^{-\frac{1}{4kt}(x^2+y^2)}$$

Example 2 : Show that $u(x,y,z) = (x^2+y^2+z^2)^{-1/2}$ satisfies Laplace's equation $u_{xx} + u_{yy} + u_{zz} = 0$ for $(x,y,z) \neq (0,0,0)$

$$u_x = -\frac{1}{2}(2x)(x^2+y^2+z^2)^{-3/2} = -x(x^2+y^2+z^2)^{-3/2}$$

$$u_{xx} = -(x^2+y^2+z^2)^{-3/2} + \frac{3}{2}x(2x)(x^2+y^2+z^2)^{-5/2}$$

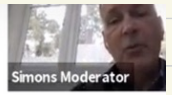
$$= -(x^2+y^2+z^2)^{-3/2} + 3x^2(x^2+y^2+z^2)^{-5/2}$$

$$u_{xx} + u_{yy} + u_{zz} = -3(x^2+y^2+z^2)^{-3/2}$$

$$+ 3(x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2) = 0$$

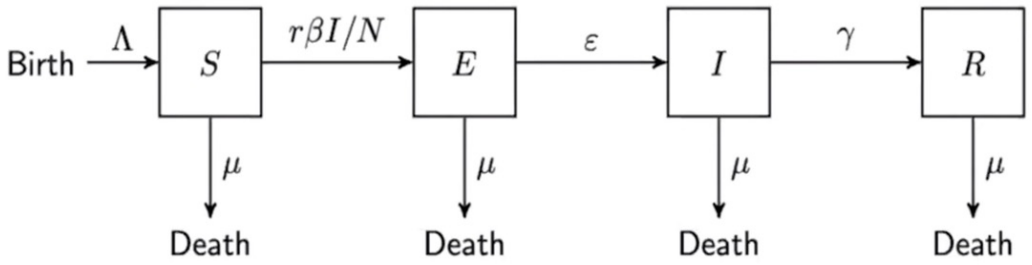
Q: What happens when $(x,y,z) = 0$?

The power of differential equations, a slide from Prof. Nicholas Jewell talk about COVID-19



SEIR Mathematical Models

Susceptible-Exposed-Infectious-Recovered Model: applicable to measles, mumps, rubella.



E: Exposed (latent) humans

ε: Per-capita rate of progression to infectious state

$$\frac{dS}{dt} = \Lambda - r\beta S\frac{I}{N} - \mu S$$
$$\frac{dE}{dt} = r\beta S\frac{I}{N} - \varepsilon E$$
$$\frac{dI}{dt} = \varepsilon E - \gamma I - \mu I$$
$$\frac{dR}{dt} = \gamma I - \mu R$$

with

$$N = S + E + I + R.$$

LONDON SCHOOL of HYGIENE &TROPICAL MEDICINE

Berkeley Public Health



SEIR Mathematical Models

$$R_0 = \left(\begin{array}{c}\text{Number of}\\ \text{contacts}\\ \text{per unit time}\end{array}\right)\left(\begin{array}{c}\text{Probability of}\\ \text{transmission}\\ \text{per contact}\end{array}\right)\left(\begin{array}{c}\text{Duration of}\\ \text{infection}\end{array}\right) \times \left(\begin{array}{c}\text{Probabililty of}\\ \text{surviving}\\ \text{exposed stage}\end{array}\right)$$

$$R_0 = r \times \beta \times \frac{1}{\gamma+\mu} \times \frac{\varepsilon}{\varepsilon+\mu} = \frac{r\beta\varepsilon}{(\gamma+\mu)(\varepsilon+\mu)}$$

- If $R_0 < 1$, the disease-free equilibrium point is globally asymptotically stable and there is no endemic equilibrium point (the disease dies out).
- If $R_0 > 1$, the disease-free equilibrium point is unstable and a globally asymptotically stable endemic equilibrium point exists.