Let $\mu$ be a Radon measure in $\mathbb{R}^m$ such that

$$0 < \theta^n(\mu, a) = \lim_{r \to 0} \frac{\mu(B(a, r))}{r^n} < \infty \text{ for } \mu - \text{a.e } a \in \mathbb{R}^m$$

Then for $\mu$-a.e. $a \in \mathbb{R}^m$ every $\nu \in \text{Tan}(\mu, a)$ is $n$-uniform with $0 \in \text{spt } \nu$.

Thus to understand the infinitesimal structure $\mu$ we need to understand the structure of $n$-uniform measures.
A measure $\mu$ on $\mathbb{R}^m$ is uniformly distributed if there is a function $f_\mu : (0, \infty) \to [0, \infty]$ such that

- $\mu(B(x, r)) = f_\mu(r)$ for all $x \in \text{spt} \mu$ and all $r > 0$.
- $f_\mu(r) < \infty$ for some $r$.

For $\mu$ uniformly distributed in $\mathbb{R}^m$, $x \in \mathbb{R}^m$ and $0 < s \leq r < \infty$

$$\mu(B(x, r)) \leq 5^m \left(\frac{r}{s}\right)^m f_\mu(s).$$

$(\star)$

Cover $B(x, r) \subset \bigcup_{i=1}^N B(z_i, s/2)$ with $|z_i - z_j| \geq s/2$. Since $\{B(z_i, s/4)\}_{i=1}^N$ are disjoint then $N \left(\frac{s}{4}\right)^m \leq \left(\frac{5}{4}r\right)^m$.

$$\mu(B(x, r)) \leq \sum_{i=1}^N \mu(B(z_i, s/2)).$$

Consider 2 cases: $\mu(B(z_i, s/2)) = 0$ or $\mu(B(z_i, s/2)) > 0$. If $\mu(B(z_i, s/2)) > 0$ there is $z \in \text{spt} \mu \cap B(z_i, s/2)$, and $B(z_i, s/2) \subset B(z, s)$. 
A measure $\mu$ on $\mathbb{R}^m$ is uniformly distributed if there is a function $f_\mu : (0, \infty) \to [0, \infty]$ such that

$\mu(B(x, r)) = f_\mu(r)$ for all $x \in \text{spt } \mu$ and all $r > 0$.

$\ f_\mu(r) < \infty$ for some $r$.

For $\mu$ uniformly distributed in $\mathbb{R}^m$, $x \in \mathbb{R}^m$ and $0 < s \leq r < \infty$

$$\mu(B(x, r)) \leq 5^m \left( \frac{r}{s} \right)^m f_\mu(s).$$

For $\mu$ uniformly distributed in $\mathbb{R}^m$ ($\star$)

$\ f_\mu(r) < \infty$ for all $r > 0$

$\int e^{-s|x-z|^2} \, d\mu(z) = \int_0^1 \mu(B(x, \sqrt{-\ln r/s})) \, dr$ converges for $s > 0$, $x \in \mathbb{R}^m$.

$\int e^{-s|x-z|^2} \, d\mu(z) = \int e^{-s|y-z|^2} \, d\mu(z)$ for all $x, y \in \text{spt } \mu$ and $s > 0$. 

Tatiana Toro (University of Washington) Structure of $n$-uniform measure in $\mathbb{R}^m$ February 17, 2016 4 / 23
Let $\mu$ be uniformly distributed in $\mathbb{R}^m$, $x_0 \in \text{spt } \mu$, for $s > 0$ and $x \in \mathbb{R}^m$

- $F(x, s) = \int \left( e^{-s|x-z|^2} - e^{-s|x_0-z|^2} \right) d\mu(z)$ is well defined and independent of $x_0$.
- $x \in \text{spt } \mu$ iff $F(x, s) = 0$ for all $s > 0$.
- If $x \notin \text{spt } \mu$ there is $s_x > 0$ so that $s > s_x$, $F(x, s) < 0$ ($\ast$).
- $\text{spt } \mu$ is a real analytic variety.
- There are $n \in \{0, 1, \ldots, m\}$, $c > 0$ and $G \subset \mathbb{R}^m$ an open set such that
  - $G \cap \text{spt } \mu$ is an $n$-dimensional analytic submanifold of $\mathbb{R}^m$.
  - $\mu(\mathbb{R}^m \setminus G) = \mathcal{H}^n(\mathbb{R}^m \setminus G) = 0$.
  - $\mathbb{R}^m \setminus G$ countable union of analytic submanifolds of $\mathbb{R}^m$ of dimension less than $n$.
  - $\mu(A) = c\mathcal{H}^n(A \cap G \cap \text{spt } \mu) = c\mathcal{H}^n(A \cap \text{spt } \mu)$ for $A \subset \mathbb{R}^m$ Borel.
Let $\mu$ be a Radon measure in $\mathbb{R}^m$ such that

$$0 < \theta^n(\mu, a) = \lim_{r \to 0} \frac{\mu(B(a, r))}{r^n} < \infty \text{ for } \mu - a.e \ a \in \mathbb{R}^m$$

Then:

- For $\mu$-a.e. $a \in \mathbb{R}^m$ every $\nu \in \text{Tan}(\mu, a)$ is $n$-uniform thus uniformly distributed.

- For $x \in G \cap \text{spt} \nu$ ($n$-analytic submanifold) if $\lambda \in \text{Tan}(\nu, x)$ then there exists $c > 0$ such that $\lambda = c\mathcal{H}^n \subseteq V_x$ where $V_x = T_x\text{spt} \nu - x$.

- Since tangents to tangents are tangents $\lambda \in \text{Tan}(\mu, a)$.

- $\lambda \in \mathcal{F}_{n,m} = \mathcal{F} = \{c\mathcal{H}^n \subseteq V : c > 0, \ V \in G(m, n)\}$, i.e. $\lambda$ is flat.

- For $\mu$-a.e. $a \in \mathbb{R}^m$, $\text{Tan}(\mu, a) \cap \mathcal{F} \neq \emptyset$. 
Are all $n$-uniform measures flat? How large can the singular set be?

**Theorem** If $\nu$ is $n$-uniform in $\mathbb{R}^{n+1}$, $\Sigma = \text{spt } \nu$ then $\nu = cH^n \subset \Sigma$ and modulo translation and rotation

- **Preiss** For $n = 1, 2$, $\Sigma = \mathbb{R}^n \times \{0\}$

- **Kowalski-Preiss** For $n \geq 3$,
  
  $\Sigma = \mathbb{R}^n \times \{0\}$, or
  $\Sigma = \{(x_1, x_2, x_3, x_4, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} : x_4^2 = x_1^2 + x_2^2 + x_3^2\}$.

Recent work of D. Nimer addresses the question in higher codimensions.
How do we prove rectifiability of \( \mu \)?

**Theorem (Preiss)** Let \( \mu \) be a Radon measure in \( \mathbb{R}^m \) with

\[
0 < \theta^*_n(\mu, a) \leq \theta^{n,*}(\mu, a) < \infty \quad \mu - \text{a.e. } a \in \mathbb{R}^m.
\]

Then the following are equivalent:

- \( \mu \) is \( n \)-rectifiable
- \( \mu - \text{a.e. } a \in \mathbb{R}^m \) there is \( V_a \) an \( n \)-dimensional space in \( \mathbb{R}^m \) so that
  \[
  \text{Tan}(\mu, a) = \{ c\mathcal{H}^n \subseteq V_a : 0 < c < \infty \}
  \]
- \( \mu - \text{a.e. } a \in \mathbb{R}^m, \text{Tan}(\mu, a) \subseteq \mathcal{F} \).

Does \( \text{Tan}(\mu, a) \cap \mathcal{F} \neq \emptyset \) \( \implies \) \( \text{Tan}(\mu, a) \subseteq \mathcal{F} \) for \( \mu \)-a.e. \( a \in \mathbb{R}^m \)?
Metric on the space of Radon measures in $\mathbb{R}^m$

For $r \in (0, \infty)$, let

$$\mathcal{L}(r) = \{ f : \mathbb{R}^m \to [0, \infty), \text{spt } f \subset B_r \text{ and } \text{Lip } f \leq 1 \}.$$ 

For Radon measures $\Phi$ and $\Psi$ on $\mathbb{R}^m$ set

$$F_r(\Phi, \Psi) = \sup \left\{ \left| \int f \, d\Phi - \int f \, d\Psi \right| ; f \in \mathcal{L}(r) \right\}.$$ 

$$F_r(\Phi) = F_r(\Phi, 0) = \int (r - |z|)^+ \, d\Phi(z). \quad (*)$$

- If $\mu, \mu_i \ (i \in \mathbb{N})$ Radon measures in $\mathbb{R}^m$, then

$$\mu_i \rightharpoonup \mu \text{ iff } \lim_{i \to \infty} F_r(\mu_i, \mu) = 0 \quad \forall r > 0.$$
Cones of measures

- Let $\mathcal{R}$ be the space of Radon measures in $\mathbb{R}^m$. For $\Phi, \Psi \in \mathcal{R}$ define
  \[
d(\Phi, \Psi) = \sum_{i=1}^{\infty} 2^{-i} \min\{1, F_i(\Phi, \Psi)\}.
  \]
  $(\mathcal{R}, d)$ is a complete separable metric space. (\*)

- If $\mu, \mu_i (i \in \mathbb{N})$ Radon measures in $\mathbb{R}^m$, then
  \[
  \mu_i \rightharpoonup \mu \text{ iff } \lim_{i \to \infty} d(\mu_i, \mu) = 0.
  \]

- $\mathcal{M} \subset \mathcal{R}$, $0 \notin \mathcal{M}$ is a cone if $c\Psi \in \mathcal{M}$ for all $\Psi \in \mathcal{M}$ and $c > 0$.

- A cone $\mathcal{M}$ is a $d$-cone if $T_{0,r}[\Psi] \in \mathcal{M}$ for all $\Psi \in \mathcal{M}$.

- For $s > 0$ the $s$-distance between a $d$-cone $\mathcal{M}$ and $\Phi \in \mathcal{R}$ is
  \[
d_s(\Phi, \mathcal{M}) = \inf \left\{ F_s\left(\frac{\Phi}{F_s(\Phi)}, \Psi\right) : \Psi \in \mathcal{M} \text{ and } F_s(\Psi) = 1 \right\}
  \]
  if $F_s(\Phi) \neq 0$. If $F_s(\Phi) = 0$ set $d_s(\Phi, \mathcal{M}) = 1$. 

A picture to illustrate $d_s(\Phi, \mathcal{M})$

\[ \{ \Psi \in \mathcal{R} : F_s(\Psi) = 1 \} \]

\[ d_s(\Phi, \mathcal{M}) = \inf \left\{ F_s\left( \frac{\Phi}{F_s(\Phi)} \right), \Psi : \Psi \in \mathcal{M} \text{ and } F_s(\Psi) = 1 \right\} \]
Additional properties

- If $\mathcal{M}$ d-cone and $\Phi \in \mathcal{R}$
  - $d_s(\Phi, \mathcal{M}) \leq 1$ (*)
  - $d_s(\Phi, \mathcal{M}) = d_1(T_{0,s}[\Phi], \mathcal{M})$

- If $\mu_i \rightharpoonup \mu$ and $F_s(\mu) > 0$ then $d_s(\mu, \mathcal{M}) = \lim_{i \to \infty} d_s(\mu_i, \mathcal{M})$.

- If $\mu$ is a non-zero Radon measure $\text{Tan}(\mu, a)$ is a d-cone. If $\nu \in \text{Tan}(\mu, a)$, $T_{0,r}[\nu] \in \text{Tan}(\mu, a)$.

- If $\mu$ is a non-zero Radon measure $\{\nu \in \text{Tan}(\mu, a) : F_1(\nu) = 1\}$ is closed under weak convergence (i.e. in the topology induced by $d$).

- The basis of a d-cone $\mathcal{M}$ is the set
  $$\{\Psi \in \mathcal{M} : F_1(\Psi) = 1\}.$$
Compact basis

Let \( \mathcal{M} \) be a d-cone in \( \mathbb{R} \) with a closed basis then \( \mathcal{M} \) has a compact basis iff there exists \( \kappa > 1 \) such that

\[
\Psi(B(0, 2r)) \leq \kappa \Psi(B(0, r))
\]

for all \( \Psi \in \mathcal{M} \) and \( r > 0 \), i.e. the doubling constant is uniform on \( \mathcal{M} \).

Let \( \mu \in \mathcal{R} \), \( a \in \text{spt} \mu \) if

\[
c_a = \limsup_{r \to 0} \frac{\mu(B(a, 2r))}{\mu(B(a, r))} < \infty,
\]

then \( \text{Tan}(\mu, a) \) has a compact basis. (\( * \))
Let $\mathcal{M}$ and $\mathcal{F}$ be $d$-cones in $\mathcal{R}$. Assume that $\mathcal{F} \subset \mathcal{M}$, $\mathcal{F}$ relatively closed in $\mathcal{M}$ and $\mathcal{M}$ has a compact basis. Suppose that there exists $\epsilon_0 > 0$ such that if

$$d_r(\Phi, \mathcal{F}) < \epsilon_0 \quad \forall r \geq r_0 > 0 \text{ then } \Phi \in \mathcal{F}.$$  

Then for a Radon measure $\mu$ and $a \in \text{spt } \mu$ if

$$\text{Tan}(\mu, a) \subset \mathcal{M} \text{ and } \text{Tan}(\mu, a) \cap \mathcal{F} \neq \emptyset \text{ then } \text{Tan}(\mu, a) \subset \mathcal{F}$$
Let $\mu$ be a Radon measure in $\mathbb{R}^m$ such that

$$0 < \theta^n(\mu, a) = \lim_{r \to 0} \frac{\mu(B(a, r))}{r^n} < \infty$$

for $\mu - a.e. \ a \in \mathbb{R}^m$.

Then for $\mu$-a.e. $a \in \mathbb{R}^m$:

- $\mathcal{F} \subset \mathcal{M}$, where $\mathcal{F}$ is the set of $n$-flat measures in $\mathbb{R}^m$ and $\mathcal{M}$ is the set of $n$-uniform measures in $\mathbb{R}^m$ with 0 in their support.

- $\mathcal{F}$ is closed in $\mathcal{M}$, which is a d-cone with compact basis.

- $\text{Tan}(\mu, a) \subset \mathcal{M}$.

- $\text{Tan}(\mu, a) \cap \mathcal{F} \neq \emptyset$.

- If (⋆) holds then $\text{Tan}(\mu, a) \subset \mathcal{F} \implies \mu$ $n$-rectifiable.
What does (∗) really mean?

Let \( \Phi \) be \( n \)-uniform. Recall

\[
d_r(\Phi, \mathcal{F}) = \inf \left\{ F_r\left( \frac{\Phi}{F_r(\Phi)} \right), \Psi : \Psi \in \mathcal{F} \text{ and } F_r(\Psi) = 1 \right\}.
\]

\[
F_r(\Phi) = \int (r - |y|)^+ d\Phi = \int \Phi(\{ y : (r - |y|)^+ > t \}) \, dt
\]

\[
= \int_0^r \Phi(\{ y : |y| < r - t \}) \, dt = \int_0^r \Phi(B_{r-t}) \, dt
\]

\[
= \int_0^r c(r - t)^n \, dt = c \frac{r^{n+1}}{n+1}
\]

- \( d_r(\Phi, \mathcal{F}) = d_r(c\Phi, \mathcal{F}) \) for \( c = \omega_n \) then \( F_r(\Phi) = \frac{\omega_n r^{n+1}}{n+1} \).
- If \( \Psi = c\mathcal{H}^n \subset V, F_r(\Psi) = c \frac{\omega_n r^{n+1}}{n+1} = 1 \) implies \( c = \left( \frac{\omega_n r^{n+1}}{n+1} \right)^{-1} \).
\[ d_r(\Phi, \mathcal{F}) = \inf_{V \in \mathcal{G}(m,n)} \frac{n+1}{\omega_n r^{n+1}} \sup_{f \in \mathcal{L}(r)} \left| \int f \, d\Phi - \int f \, d\mathcal{H}^n \mathbf{1}_V \right| \]

Note that for
\[ f(x) = \text{dist}^2(x, V) \frac{(r - |x|)^+}{r^2} \in \mathcal{L}(r). \]

\[
\frac{n+1}{\omega_n r^{n+1}} \left| \int f \, d\Phi - \int f \, d\mathcal{H}^n \mathbf{1}_V \right| \geq \frac{n+1}{\omega_n r^{n+1}} \int f \, d\Phi
\]
\[
\geq \frac{n+1}{\omega_n r^{n+1}} \int_{B_{r/2}} \text{dist}^2(x, V) \frac{r}{2r^2} \, d\Phi
\]
\[
\geq \frac{n+1}{2\omega_n r^{n+2}} \int_{B_{r/2}} \text{dist}^2(x, V) \, d\Phi
\]

If (*) \( d_r(\Phi, \mathcal{F}) < \epsilon_0 \) for \( r \geq r_0 > 0 \), then
\[
\inf_{V \in \mathcal{G}(m,n)} \frac{n+1}{\omega_n r^{n+2}} \int_{B_{r/2}} \text{dist}^2(x, V) \, d\Phi < 2\epsilon_0.
\]
The condition
\[ d_r(\Phi, \mathcal{F}) < \epsilon_0 \quad \forall r \geq r_0 \]
implies that there exists \( \epsilon_1 > 0 \) such that
\[
\inf_{V \in G(m, n)} \frac{1}{\omega_n r^{n+2}} \int_{B_r} \text{dist}^2(x, V) \, d\Phi < \epsilon_1 \quad \text{for} \quad r \geq r_0.
\]

Thus for \( r \geq r_0 \) there exists \( V_r \in G(m, n) \) such that
\[
(\spadesuit) \quad \frac{1}{\omega_n r^{n+2}} \int_{B_r} \text{dist}^2(x, V_r) \, d\Phi \leq \epsilon_1
\]

To prove \((\star)\) we need to show that if \( \Phi \) \( n \)-uniform is “close to flat at infinity” in \( L^2 \) as in \((\spadesuit)\) then \( \Phi \) is flat.
There is a large good subset $G_r$ of spt $\Phi$ in $B_r$ which is close to $V_r$.

By Chebyshev’s inequality if (♠) holds then

$$G_r = \{ x \in \text{spt } \Phi \cap B_r : \text{dist } (x, V_r) \leq \epsilon_1^{1/4} r \}$$

satisfies

$$\Phi(B_r \setminus G_r) \leq \epsilon_1^{1/2} \Phi(B_r).$$
The small subset of \( \text{spt} \Phi \) in \( B_r \setminus G_r \) is also close to \( V_r \)

Let \( r_1 = r(1 - 2\varepsilon_1^{1/2n}) \). If \( y \in (B_{r_1} \setminus G_r) \cap \text{spt} \Phi \),

\[
\rho = \min\{\text{dist} (y, G_r), \text{dist} (y, \partial B_r)\} > 0 \text{ then } B_{\rho}(y) \subset B_r \setminus G_r
\]

\[
\omega_n \rho^n = \Phi(B_{\rho}(y)) \leq \Phi(B_r \setminus G_r) \leq \varepsilon_1^{1/2} \Phi(B_r) = \omega_n \varepsilon_1^{1/2} r^n.
\]

Thus \( \rho = \text{dist} (y, G_r) \leq \varepsilon_1^{1/2n} r \). For \( y \in (B_{r(1-2\varepsilon_1^{1/2n})} \setminus G_r) \cap \text{spt} \Phi \),

\[
\text{dist} (y, V_r) \leq \text{dist} (y, G_r) + \text{dist} (G_r, V_r) \leq \varepsilon_1^{1/2n} r.
\]
If \( d_r(\Phi, \mathcal{F}) < \epsilon_0 \) for \( r \geq r_0 > 0 \) there is \( V_r \) such that

\[
(\spadesuit) \quad \frac{1}{\omega_n r^{n+2}} \int_{B_r} \text{dist}^2(x, V_r) \, d\Phi \leq \epsilon_1
\]

with \( \epsilon_1 = C(n)\epsilon_0 \).

In this case for \( y \in B_{r(1-2\epsilon_1^{1/2n})} \cap \text{spt} \, \Phi \), \( \text{dist} (y, V_r) \leq \epsilon_1^{1/2n} r \).

\( \text{spt} \, \Phi \) close to \( V_r \) in \( L^2 \) sense in \( B_r \) then \( \text{spt} \, \Phi \) close \( V_r \) in \( L^\infty \) sense in \( B_{r(1-2\epsilon_1^{1/2n})} \).
(♣) yields additional geometric information for Φ

Not only is spt Φ close to $V_r$ but $V_r$ is also close to spt Φ.

Can there be holes of size $τ$?
NO if $ε_1$ is small enough.
Lemma: Given \( \tau > 0 \) there exists \( \epsilon_1 = \epsilon_1(\tau, n, m) > 0 \) such that if \( \Phi \) is \( n \)-uniform in \( \mathbb{R}^m \) with \( 0 \in \text{spt} \Phi \), \( \Phi(B_1) = 1 \) and for some \( V \in G(m, n) \)

\[
\int_{B_1} \text{dist}^2(x, V) \, d\Phi < \epsilon_1,
\]

then for all \( z \in V \cap B_1 \)

\[
B(z, \tau) \cap \text{spt} \Phi \neq \emptyset.
\]