Rectifiability of sets and measures

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A non-zero Radon measure $\nu$ is a tangent measure to $\mu$ at $a \in \mathbb{R}^m$ if there exist sequences $\{r_i\}_i, r_i \to 0$ and $\{c_i\}_i, c_i > 0$ such that

$$c_i T_{a,r_i}[\mu] \rightharpoonup \nu \text{ as } i \to \infty,$$

i.e. $\forall \varphi \in C_c(\mathbb{R}^m)$

$$\lim_{i \to \infty} c_i \int \varphi \left( \frac{x-a}{r_i} \right) d\mu(x) = \int \varphi \, d\nu.$$

The set of all tangent measures to $\mu$ at $a$ is denoted by

$$\text{Tan}(\mu,a).$$
Preiss (1987) For any Radon measure $\mu$, $\text{Tan}(\mu, a) \neq \emptyset$, $\mu$- a.e. $a \in \mathbb{R}^m$.

Let $\mu$ be a Radon measure in $\mathbb{R}^m$, if for $a \in \mathbb{R}^m$

$$c_a = \limsup_{r \to 0} \frac{\mu(B(a, 2r))}{\mu(B(a, r))} < \infty \quad (\mu \text{ asymptotically doubling at } a)$$

then every sequence $\{r_i\}_i$, $r_i \to 0$ contains a subsequence $\{r_{i_k}\}_{i_k}$ such that

$$\mu_{a, r_{i_k}} = \frac{1}{\mu(B(a, r_{i_k}))} T_{a, r_{i_k}}[\mu] \rightharpoonup \nu \neq 0 \quad \text{and} \quad \nu \in \text{Tan}(\mu, a).$$

Moreover any tangent measure is obtained this way up to a constant.
Proof

Since
\[ 1 \leq c_a = \inf_{s>0} \sup_{0<r<s} \frac{\mu(B(a,2r))}{\mu(B(a,r))} < \infty \]

there is \( s > 0 \) such that
\[ \sup_{0<r<s} \frac{\mu(B(a,2r))}{\mu(B(a,r))} \leq 2c_a. \]

Let \( \mu_i = \frac{1}{\mu(B(a,r_i))} T_{a,r_i} [\mu] \). Given a compact set \( K \subset \mathbb{R}^m \), \( K \subset B(0,2^\ell) \) for some \( \ell \). Then for \( i \) large enough
\[ \mu_i(K) \leq \mu_i(B(0,2^\ell)) = \frac{\mu(B(a,2^\ell r_i))}{\mu(B(a,r_i))} \leq (2c_a)^\ell. \]

Thus \( \sup_i \mu_i(K) < \infty \). By compactness there exists a subsequence such that \( \mu_{i_k} \rightharpoonup \nu \) where \( \nu \) is a Radon measure.
Proof

- \( \nu \neq 0 \) and \( 0 \in \text{spt} \nu \): for \( j \in \mathbb{N} \)

\[
\nu(B(0, 2^{-j})) \geq \nu(B(0, 2^{-j-1})) \geq \limsup_{i_k \to \infty} \mu_{i_k}(B(0, 2^{-j-1}))
\]

\[
\geq \limsup_{i_k \to \infty} \frac{\mu(B(a, 2^{-j-1}r_i))}{\mu(B(a, r_i))}
\]

\[
\geq (2c_a)^{-j-1} > 0.
\]

- Let \( \lambda \in \text{Tan}(\mu, a) \) then \( c_i T_{a, r_i}[\mu] \rightharpoonup \lambda \) for some \( c_i > 0 \) and \( r_i \to 0 \).

Note that

\[
c_i T_{a, r_i}[\mu](B(0, 1)) = c_i \mu(B(a, r_i)) \sim 1 \text{ uniformly on } i.
\]

Passing to a subsequence \( c_{i_k} \mu(B(a, r_{i_k})) \to c_0 > 0 \) and

\[
\frac{c_0}{\mu(B(a, r_{i_k}))} T_{a, r_{i_k}}[\mu] \rightharpoonup \lambda.
\]
Remarks

If for $s > 0$

$$\theta^s_*(\mu, a) = \liminf_{r \to 0} \frac{\mu(B(a, r))}{r^s} > 0$$

$$\theta^{*,s}_*(\mu, a) = \limsup_{r \to 0} \frac{\mu(B(a, r))}{r^s} < \infty,$$

then $\mu$ asymptotically doubling at $a$. Note

$$\frac{\mu(B(a, 2r))}{\mu(B(a, r))} = 2^s \frac{\mu(B(a, 2r))}{(2r)^s} \cdot \frac{r^s}{\mu(B(a, r))}.$$ 

$$\limsup_{r \to 0} \frac{\mu(B(a, 2r))}{\mu(B(a, r))} \leq 2^s \frac{\theta^{*,s}_*(\mu, a)}{\theta^s_*(\mu, a)}.$$ 

For $\lambda \in \text{Tan}(\mu, a)$ there exists $c_0 > 0$ and $r_i \to 0$ such that

$$\frac{c_0}{\mu(B(a, r_i))} T_{a, r_i} [\mu] \to \lambda$$

and

$$\frac{\mu(B(a, r_i))}{r_i^s} \to c_1.$$ 

Hence

$$c_0 c_1^{-1} r_i^{-s} T_{a, r_i} [\mu] \to \lambda.$$
Theorem (Preiss) Let $s > 0$, $\mu$ a Radon measure in $\mathbb{R}^m$. Let

$$A = \{ a \in \mathbb{R}^m : 0 < \theta_s^* (\mu, a) \leq \theta^{s,*} (\mu, a) < \infty \}$$

then for $\mu$-a.e. $a \in A$ and every $\nu \in \text{Tan}(\mu, a)$ there is $c > 0$ such that for all $x \in \text{spt} \nu$ and $r > 0$

$$tcr^s \leq \nu (B(x, r)) \leq cr^s$$

where

$$t = t(a) = \frac{\theta^s (\mu, a)}{\theta^{s,*} (\mu, a)}.$$
Proof

- \( A = \bigcup_{j=1}^{\infty} A_j \) such that \( \theta^s_*(\mu, \cdot) \) and \( \theta^{s,*}(\mu, \cdot) \) oscillate very little.

- Given \( \epsilon > 0 \) decompose each \( A_i \) into a countable collection so that there are \( B_i \subset A \) and \( \tau_i, c_i, r_i > 0 \) such that \( \mu(A \setminus \bigcup_{i}^{\infty} B_i) = 0 \) and for \( a \in B_i \) and \( 0 < r < r_i \)

\[
\tau_i \leq t(a) \leq \tau_i + \epsilon \quad \& \quad \tau_i c_i r^s \leq \mu(B(a, r_i)) \leq c_i r^s.
\]

- If for all \( a \in B \) and \( r \leq r_0 \)

\[
\tau c r^s \leq \mu(B(a, r)) \leq c r^s.
\]

Then if \( a \) is a \( \mu \)-density point of \( B \), and \( \nu \in \text{Tan}(\mu, a) \) with \( \nu = \lim_{r_i \to 0} r_i^{-s} T_{a, r_i}[\mu] \) then for all \( x \in \text{spt} \nu \) and \( r > 0 \)

\[
\tau c r^s \leq \nu(B(x, r)) \leq c r^s.
\]
Definition: Let \( s > 0 \). A Radon measure \( \nu \) in \( \mathbb{R}^m \) is \( s \)-uniform if there is \( c > 0 \) such that for all \( x \in \text{spt} \nu \) and \( r > 0 \)

\[
\nu(B(x, r)) = cr^s.
\]

Example: \( \mathcal{H}^m \) in \( \mathbb{R}^m \). Are there others?
Find an example of a measure \( \nu \) in \( \mathbb{R}^3 \) such that \( \nu(B(x, r)) = \pi r^2 \) for all \( x \in \text{spt} \nu \) and \( 0 < r < 1 \).

Corollary (Preiss) Let \( s > 0 \), \( \mu \) a Radon measure in \( \mathbb{R}^m \). Let

\[
A = \{ a \in \mathbb{R}^m : 0 < \theta^s(\mu, a) < \infty \}
\]

then for \( \mu \)-a.e. \( a \in A \) and every \( \nu \in \text{Tan}(\mu, a) \), \( \nu \) is \( s \)-uniform with \( 0 \in \text{spt} \nu \).
Theorem (Marstrand) (1954) Let $s > 0$, $\mu$ a Radon measure in $\mathbb{R}^m$. Let
\[ A = \{ a \in \mathbb{R}^m : 0 < \theta^s(\mu, a) < \infty \}. \]
If $\mu(A) > 0$ then $s \in \mathbb{N}$.

Proof:
- For $\mu$-a.e. $a \in A$ and every $\Phi \in \text{Tan}(\mu, a)$, $\Phi$ is $s$-uniform with $0 \in \text{spt} \Phi$.
- The theorem reduces to show that if $\Phi$ is a non-zero $s$-uniform measure in $\mathbb{R}^m$ then $s \in \mathbb{N}$.
Contradiction argument

- Let $\Phi \neq 0$ be $s$-uniform in $\mathbb{R}^m$ with $s \not\in \mathbb{N}$ and $0 \in \text{spt } \Phi$.
- Let $m$ be the smallest dimension where this happens. By a dimension reduction argument there exists an $s$-uniform measure in $\mathbb{R}^{m-1}$.
- Since $s < m$, $\text{spt } \Phi \neq \mathbb{R}^m$ (Check this using a covering argument).
Moreover since $\text{spt } \Phi$ is a closed set there exist $z \in \mathbb{R}^m \setminus \text{spt } \Phi$ and $r > 0$ such that $B(z, r) \cap \text{spt } \Phi = \emptyset$ but $y_0 \in \partial B(z, r) \cap \text{spt } \Phi$. 

![Diagram](https://via.placeholder.com/150)
There exists $\nu \in \text{Tan}(\Phi, y_0)$, $\nu$ $s$-uniform.

The blow up of $\overline{B(z, r)}$ at $y_0$ is the half space $\{x : \langle x, e \rangle \leq 0\}$ where $e = \frac{y_0 - z}{|y_0 - z|}$ and $\text{spt} \nu \subset \{x : \langle x, e \rangle \geq 0\}$.

Let $B_r = B_r(0)$ consider the center of mass of $\nu$ on $B_r$ is

$$b_r = \frac{1}{\nu(B_r)} \int_{B_r} z \, d\nu.$$
The center of mass - First moment estimate

- If \( b_r = 0 \) for all \( r > 0 \) then

\[
0 = \langle b_r, e \rangle = \frac{1}{\nu(B_r)} \int_{B_r} \langle z, e \rangle \, d\nu \geq 0 \implies \langle z, e \rangle = 0 \text{ for all } z \in \text{spt} \, \nu.
\]

Then \( \text{spt} \, \nu \subset \langle e \rangle^\perp \) an \((m-1)\)-dimensional space: contradiction.

- If \( b_r \neq 0 \) for some \( r > 0 \) we estimate

\[
2\langle b_r, y \rangle = \frac{2}{\nu(B_r)} \int_{B_r} \langle x, y \rangle \, d\nu(x)
\]

for \( y \in \text{spt} \, \nu \).

**First moment estimate:** For \( y \in \text{spt} \, \nu \cap B_{r/2} \) since \( 0 \in \text{spt} \, \nu \) we have

\[
|2\langle b_r, y \rangle| \leq C(r)|y|^2
\]
For \( y \in \text{spt} \nu \) since \( 0 \in \text{spt} \nu \)

\[ 2\langle x, y \rangle = (r^2 - |x - y|^2) - (r^2 - |x|^2) + |y|^2 \]

\[ \frac{2}{\nu(B_r)} \int_{B_r} \langle x, y \rangle \, d\nu(x) = \]

\[ |y|^2 + \frac{1}{\nu(B_r)} \int_{B_r} (r^2 - |x - y|^2) \, d\nu(x) - \frac{1}{\nu(B_r)} \int_{B_r} (r^2 - |x|^2) \, d\nu(x) \]

\[ \int_{B_r} (r^2 - |x|^2) \, d\nu(x) = \int_{B_r(y)} (r^2 - |x - y|^2) \, d\nu(x) \]
For \( y \in \text{spt} \nu \) since \( 0 \in \text{spt} \nu \\

\[
\int_{B_r(y)} (r^2 - |x - y|^2) \, d\nu(x) = \int_0^{r^2} \nu(\{x : (r^2 - |x - y|^2 > t\}) \, dt \\
= \int_0^{r^2} \nu(B(y, \sqrt{t - r^2})) \, dt \\
= \int_0^{r^2} \nu(B(0, \sqrt{t - r^2})) \, dt \\
= \int_{B_r} (r^2 - |x|^2) \, d\nu(x)
\]
For $y \in \text{spt} \nu \cap B_{r/2}$ since $0 \in \text{spt} \nu$

- $2 \langle x, y \rangle = |y|^2 + (r^2 - |x - y|^2) - (r^2 - |x|^2)$
- $\int_{B_r} (r^2 - |x|^2) \, d\nu(x) = \int_{B_r(y)} (r^2 - |x - y|^2) \, d\nu(x)$
- $|2 \langle b_r, y \rangle| \leq |y|^2 + \frac{1}{\nu(B_r)} \left| \int_{(B_r(y) \setminus B_r) \cup (B_r \setminus B_r(y))} (r^2 - |x - y|^2) \, d\nu(x) \right|$
  - For $x \in (B_r(y) \setminus B_r) \subset (B_{r+|y|} \setminus B_r)$, $0 \leq r^2 - |x - y|^2 \leq 3r|y|$  
  - For $x \in (B_r \setminus B_r(y)) \subset (B_r \setminus B_{r-|y|})$, $0 \leq |x - y|^2 - r^2 \leq 3r|y|$
- $|2 \langle b_r, y \rangle| \leq |y|^2 + \frac{3r|y|}{\nu(B_r)} [\nu(B_{r+|y|} \setminus B_r) + \nu(B_r \setminus B_{r-|y|})]$
Quadratic estimate

For $y \in \text{spt} \, \nu \cap B_{r/2}$ since $0 \in \text{spt} \, \nu$

- $2\langle x, y \rangle = |y|^2 + (r^2 - |x - y|^2) - (r^2 - |x|^2)$

- $|2 \langle b_r, y \rangle| \leq |y|^2 + \frac{3r|y|}{\nu(B_r)}[\nu(B_{r+|y|} \setminus B_r) + \nu(B_r \setminus B_{r-|y|})]$  

- $|2 \langle b_r, y \rangle| \leq |y|^2 + \frac{3r|y|}{\nu(B_r)}[\nu(B_{r+|y|} \setminus B_r-|y|)]$  

- $|2 \langle b_r, y \rangle| \leq |y|^2 + \frac{6r|y|}{\nu(B_r)}[(r + |y|)^s - (r - |y|)^s]$
For $y \in \text{spt } \nu \cap B_{r/2}$ since $0 \in \text{spt } \nu$

- $2\langle x, y \rangle = |y|^2 + (r^2 - |x - y|^2) - (r^2 - |x|^2)$

- $\int_{B_r} (r^2 - |x|^2) \, d\nu(x) = \int_{B(y, r)} (r^2 - |x - y|^2) \, d\nu(x)$

- $|2\langle b_r, y \rangle| \leq |y|^2 + \frac{3r|y|}{\nu(B_r)}[(r + |y|)^s - (r - |y|)^s]$

- $|2\langle b_r, y \rangle| \leq |y|^2 + 6\frac{r|y|}{r^s}[(r + |y|)^s - (r - |y|)^s]$

- $|2\langle b_r, y \rangle| \leq C(r)|y|^2$
Another blow up

- For $y \in \text{spt} \, \nu \cap B_{r/2}$ since $0 \in \text{spt} \, \nu$ we have

  $$|2 \langle b_r, y \rangle| \leq C(r)|y|^2 \implies |2 \langle b_r, \frac{y}{r_i} \rangle| \leq C(r) r_i |\frac{y}{r_i}|^2$$

- Let $\lambda \in \text{Tan}(\nu, 0)$ then $\lambda$ is $s$-uniform. For $\eta > 0$ and $R > 1$

  $$\lambda(\{y \in B_R : \langle y, b_r \rangle > \eta |y|\}) = 0.$$  

$$\begin{align*}
\lambda(G) & \leq \liminf_{i \to \infty} r_i^{-s} T_{0, r_i}[\nu](G) \\
& \leq \liminf_{i \to \infty} r_i^{-s} \nu(\{y \in B_{R r_i} : \langle y, b_r \rangle > \eta |y|\}) \\
& \leq \liminf_{i \to \infty} r_i^{-s} \nu(\{y \in B_{R r_i} : \eta |y| \leq C |y|^2\}) \\
& \leq \liminf_{i \to \infty} r_i^{-s} \nu(\{y \in B_{R r_i} : \eta \leq C |y|\}) = 0
\end{align*}$$

Thus $\text{spt} \, \lambda \subset \{z \in \mathbb{R}^m : \langle z, b_r \rangle = 0\}$: contradiction.
Summary of the contradiction argument

- Since the $s$ density exists $\mu$-a.e, for $\mu$-a.e. $a \in A$ and every $\Phi \in \text{Tan}(\mu, a)$, $\Phi$ is $s$-uniform with $0 \in \text{spt } \Phi$.

- Let $m$ be the smallest dimension where there is $\Phi \neq 0$ $s$-uniform in $\mathbb{R}^m$ with $s \not\in \mathbb{N}$.

- By blow-up argument there exists $\nu$ $s$-uniform with support in a half plane.

- Study of the first moment at 0 and possibly an additional blow-up yields an $s$-uniform measure supported in an $(m - 1)$-dimensional space.

- Contradiction
Let $\mu$ be a Radon measure such that the density $\theta^s(\mu, a) \in (0, \infty)$ $\mu$-a.e. in $\mathbb{R}^m$ then $\mu$-a.e. $a \in \mathbb{R}^m$ every $\nu \in \text{Tan}(\mu, a)$, $\nu$ is $s$-uniform with $0 \in \text{spt} \nu$.

The moments play an important role in understanding the structure of $n$-uniform measures.

Studying the tangent measures to a tangent measure can be useful.
Theorem (Preiss): Let $\mu$ be a Radon measure in $\mathbb{R}^m$, $\mu$- a.e. $a \in \mathbb{R}^m$ all $\nu \in \text{Tan}(\mu, a)$ satisfy

- $T_{x, \rho}[\nu] \in \text{Tan}(\mu, a)$ for all $x \in \text{spt} \nu$ and all $\rho > 0$.
- $\text{Tan}(\nu, x) \subset \text{Tan}(\mu, a)$ for all $x \in \text{spt} \nu$.

Remark: If $\mu$ is asymptotically doubling at $a$, then for all $\nu \in \text{Tan}(\mu, a)$, $0 \in \text{spt} \nu$ and in particular for all $\rho > 0$, $T_{0, \rho}[\nu] \in \text{Tan}(\mu, a)$.