

An example from the past

Let $(X, \mathcal{M}, \mu)$ be a measure space. If $f \in \mathcal{L}^+$ then

$$\nu(E) = \int_E f d\mu \quad \text{defined for } E \in \mathcal{M}$$

is also measure.

Moreover if $\mu(E) = 0$ then $\nu(E) = 0$.

Definition: A **signed measure** ν on a measurable space $(X, \mathcal{M})$ is a mapping that satisfies:

- $-\infty < \nu(E) \leq \infty$ for all $E \in \mathcal{M}$.
- If $\{E_j\}_{j=1}^{\infty}$ are disjoint subsets of $\mathcal{M}$, then

$$\nu \left(\bigcup_{j=1}^{\infty} E_j \right) = \sum_{j=1}^{\infty} \nu(E_j).$$

Example:

Let $(X, \mathcal{M}, \mu)$ be a measure space. If f is measurable and $\int f^- d\mu < \infty$ then

$$\nu(E) = \int_E f d\mu \quad \text{defined for } E \in \mathcal{M}$$

is a signed measure.

Notation: Such a function f is said to be extended μ -integrable.

Definition: Let $(X, \mathcal{M})$ be a measure space, let ν be a signed measure. The **total variation** of ν is given by

$$|\nu|(E) = \sup \sum_{j=1}^{\infty} |\nu(E_j)|,$$

where the supremum is taken over all partitions of E , i.e. all disjoint countable unions $E = \cup_{j=1}^{\infty} E_j$ with $E_j \in \mathcal{M}$.

Properties

Proposition: The total variation $|\nu|$ of a signed measure ν is itself a (positive) measure that satisfies $\nu \leq |\nu|$.

Positive variation:

$$\nu^+ = \frac{1}{2}(|\nu| + \nu)$$

Negative variation:

$$\nu^- = \frac{1}{2}(|\nu| - \nu)$$

Note: $\nu^\pm$ are measures and

$$\nu = \nu^+ - \nu^- \quad \text{and} \quad |\nu| = \nu^+ + \nu^-.$$

If $\nu(E) = \infty$, then $|\nu|(E) = \infty$ and by definition $\nu^-(E) = 0$.

- A signed measure ν is σ -**finite** if $|\nu|$ is σ -finite. In this case $\nu^\pm$ are also σ -finite.
- ν is supported on $A \in \mathcal{M}$ is

$$\nu(E) = \nu(A \cap E) \quad \text{for all } E \in \mathcal{M}.$$

Mutually singular and absolutely continuous measures

- ν and μ signed measure on $(X, \mathcal{M})$ are **mutually singular**, $\nu \perp \mu$, if there exists $A, B \in \mathcal{M}$, $A \cap B = \emptyset$

$$\nu(E) = \nu(A \cap E) \quad \text{and} \quad \mu(E) = \mu(B \cap E) \quad \text{for all} \quad E \in \mathcal{M}.$$

- If ν is a signed measure and μ is a (positive) measure on $(X, \mathcal{M})$, ν is **absolutely continuous** with respect to μ , $\nu \ll \mu$, if

$$\nu(E) = 0 \quad \text{whenever} \quad E \in \mathcal{M} \quad \text{and} \quad \mu(E) = 0.$$

Proposition: Let ν be a signed measure and let μ be a (positive) measure on $(X, \mathcal{M})$. If $|\nu|$ is finite then $\nu \ll \mu$ if and only if

$$\begin{aligned} &\text{For each } \epsilon > 0 \text{ there is } \delta > 0 \text{ such that} \\ &|\nu|(E) < \epsilon \text{ whenever } \mu(E) < \delta. \end{aligned} \tag{1}$$

In general (1) implies that $\nu \ll \mu$.

Lebesgue - Radon - Nikodym Theorem

Theorem: Suppose μ is a σ -finite positive measure, and ν is a signed σ -finite measure on $(X, \mathcal{M})$. Then there exist unique signed σ -finite measures ν_a and ν_s on $(X, \mathcal{M})$ such that

$$\nu_a \ll \mu, \nu_s \perp \mu \quad \text{and} \quad \nu = \nu_a + \nu_s.$$

Moreover $d\nu_a = f d\mu$, i.e.

$$\nu_a(E) = \int_E f(x) d\mu(x),$$

for some extended μ -integrable function f .

Definition: A Borel measure ν in $\mathbb{R}^d$ is **regular** if

- $\nu(K) < \infty$ for every compact K ;
- $\nu(E) = \inf\{\nu(U) : U \text{ open, } E \subset U\}$ for every $E \in \mathcal{B}_{\mathbb{R}^d}$.

Theorem: Let ν be a Borel regular measure in $\mathbb{R}^d$, and let $d\nu = f \, dm + d\nu_s$ be its Lebesgue-Radon-Nikodym representation. Then for m -a.e. $x \in \mathbb{R}^d$,

$$\lim_{r \rightarrow 0} \frac{\nu(B(x, r))}{m(B(x, r))} = f(x).$$