CONSTRUCTION OF THE REAL NUMBERS

We present a brief sketch of the construction of $\mathbb{R}$ from $\mathbb{Q}$ using Dedekind cuts. This is the same approach used in Rudin’s book *Principles of Mathematical Analysis* (see Appendix, Chapter 1 for the complete proof). The elements of $\mathbb{R}$ are some subsets of $\mathbb{Q}$ called cuts. On the collection of these subsets, i.e. on $\mathbb{R}$, we define an order, an addition, and a multiplication. We show that $\mathbb{R}$ endowed with this relation and these two operations is an ordered field. Each rational number can be identified with a specific cut, in such a way that $\mathbb{Q}$ can be viewed as a subfield of $\mathbb{R}$.

**Step 1.** A subset $\alpha$ of $\mathbb{Q}$ is said to be a cut if:

1. $\alpha$ is not empty, $\alpha \neq \mathbb{Q}$.
2. If $p \in \alpha$, $q \in \mathbb{Q}$, and $q < p$, then $q \in \alpha$.
3. If $p \in \alpha$, then $p < r$ for some $r \in \alpha$.

**Remarks:**

- 3 implies that $\alpha$ has no largest number.
- 2 implies that:
  - If $p \in \alpha$ and $q \notin \alpha$ then $p < q$.
  - If $r \notin \alpha$ and $r < s$ then $s \notin \alpha$.

**Example:** Let

$$\alpha = \{p \in \mathbb{Q} : p < 0\} \cup \{p \in \mathbb{Q} : p \geq 0 \text{ and } p^2 < 2\}.$$ 

Note that $\alpha$ is a cut. In fact:

1. $\alpha \subset \mathbb{Q}$, $1 \in \alpha$ thus $\alpha \neq$, and $2 \notin \alpha$ thus $\alpha \neq \mathbb{Q}$.

2. If $p \in \alpha$, $q \in \mathbb{Q}$, and $q < p$, then either $q \leq 0$ and so $q \in \alpha$, or $q > 0$ which implies $p > 0$. But since $p \in \alpha$, then $p^2 < 2$. Since $0 < q < p$ then $q^2 < p^2$. Therefore $q^2 < 2$, i.e. $q \in \alpha$. 

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3. If \( p \in \alpha \), either \( p \leq 0 \) or \( p > 0 \). If \( p \leq 0 \) then \((3)\) is satisfied with \( r = 1 \). If \( p > 0 \) and \( p^2 < 2 \) then (as shown in class) \( r = \frac{2(p+1)}{p+2} \) satisfies \( 0 < p < r \) and \( r^2 < 2 \). Thus \( r \in \alpha \), and \((3)\) also holds in this case.

**Step 2.** Let \( \mathbb{R} \) be the collection of all cuts of \( \mathbb{Q} \). For \( \alpha, \beta \in \mathbb{R} \) define \( \alpha < \beta \) to mean \( \alpha \) is a proper subset of \( \beta \), (i.e. \( \alpha \subset \beta \) but \( \alpha \neq \beta \)). \( \mathbb{R} \) is an ordered set with relation \(<\) defined above.

**Step 3.** *The ordered set \( \mathbb{R} \) has the least-upper-bound property.* Let \( A \) be a nonempty subset of \( \mathbb{R} \) which is bounded above. Let \( \gamma = \bigcup_{\alpha \in A} \alpha \). Then \( \gamma \in \mathbb{R} \) (i.e. \( \gamma \) satisfies (1), (2) and (3)), and \( \gamma = \sup A \).

**Step 4.** If \( \alpha, \beta \in \mathbb{R} \) define

\[
\alpha + \beta = \{ r + s : r \in \alpha \text{ and } s \in \beta \},
\]

\[
0^* = \{ p \in \mathbb{Q} : p < 0 \},
\]

and

\[
\alpha^* = \{ p \in \mathbb{Q} : \text{there exits } r > 0 \text{ such that } -p - r \not\in \alpha \}.
\]

\( \alpha + \beta \), \( 0^* \) and \( \alpha^* \) are cuts. The axioms for addition hold in \( \mathbb{R} \), with \( 0^* \) playing the role of 0, and \( \alpha^* \) playing the role of \(-\alpha\).

**Step 5.** After proving that the axioms of addition hold in \( \mathbb{R} \) for the operation defined in Step 4, one can show using the cancellation law that

If \( \alpha, \beta, \gamma \in \mathbb{R} \) and \( \beta < \gamma \), then \( \alpha + \beta < \alpha + \gamma \).

**Step 6.** Initially we define multiplication for positive real numbers. Let \( \mathbb{R}^+ = \{ \alpha \in \mathbb{R} : \alpha > 0^{ast} \} \). If \( \alpha, \beta \in \mathbb{R}^+ \) define

\[
\alpha \beta = \{ p \in \mathbb{Q} : p \leq rs \text{ for some } r \in \alpha, \ s \in \beta, \ r > 0, \ s > 0 \},
\]

\[
1^* = \{ p \in \mathbb{Q} : p < 1 \},
\]

and

\[
\alpha_* = \{ p \in \mathbb{Q} : p \leq 0 \} \cup \{ p \in \mathbb{Q} : p > 0 \text{ and there exits } r > 0 \text{ such that } \frac{1}{p} - r \not\in \alpha \}.
\]

\( \alpha \beta \), \( 1^* \) and \( \alpha_* \) are cuts. The axioms for multiplication hold in \( \mathbb{R}^+ \), with \( 1^* \) playing the role of 1, and \( \alpha_* \) playing the role of \( \frac{1}{\alpha} \), for \( \alpha > 0^* \). Note that if \( \alpha > 0^* \) and \( \beta > 0^* \) then \( \alpha \beta > 0^* \). One also checks that the distributive law holds in \( \mathbb{R}^+ \).
Step 7. We complete the definition of multiplication by setting $\alpha 0^* = 0^* \alpha = 0^*$, and by setting

$$\alpha \beta = \begin{cases} (-\alpha)(-\beta) & \text{if } \alpha < 0^*, \beta < 0^*, \\ -\left(\alpha \beta\right) & \text{if } \alpha < 0^*, \beta > 0^*, \\ -\left[\alpha (-\beta)\right] & \text{if } \alpha > 0^*, \beta < 0^*. \end{cases}$$

The products on the right were defined in Step 6. Having checked the axioms of multiplication in $\mathbb{R}^+$ it is simple to prove them in $\mathbb{R}$ by repeated applications of the identity $\gamma = -(-\gamma)$. The proof of the distributive law is done by cases.

*THIS COMPLETES THE SKETCH OF THE PROOF THAT $\mathbb{R}$ IS AN ORDERED FIELD WITH THE LEAST-UPPER-BOUND PROPERTY.*

Step 8. We associate with each $r \in \mathbb{Q}$ the set

$$r^* = \{ p \in \mathbb{Q} : p < r \}.$$ 

$r^*$ is a (rational) cut, thus $r^* \in \mathbb{R}$. The rational cuts satisfy the following relations:

- $r^* + s^* = (r + s)^*$.
- $r^* s^* = (rs)^*$.
- $r^* < s^*$ if and only if $r < s$.

Step 9. Step 8 says that the rational numbers can be identified with the rational cuts. This identification preserves sums, products and order. Thus the ordered field $\mathbb{Q}$ is *isomorphic* to the ordered field $\mathbb{Q}^* \subset \mathbb{R}$ whose elements are the rational cuts.

*THIS IDENTIFICATION OF $\mathbb{Q}$ WITH $\mathbb{Q}^*$ ALLOWS US TO REGARD $\mathbb{Q}$ AS A SUBFIELD OF $\mathbb{R}$.*