Lecture notes on rectifiable sets, densities, and tangent measures

Camillo De Lellis

Camillo De Lellis, Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland

E-mail address: delellis@math.unizh.ch

Contents

CHAPTER 1

Introduction

These notes are taken from the final part of a class on rectifiability given at the University of Zürich during the summer semester 2004. The main aim is to provide a self–contained reference for the proof of the following remarkable theorem

THEOREM 1.1. *Let μ be a locally finite measure on $\mathbb{R}^n$ and α a nonnegative real number. Assume that the following limit exists, is finite and non–zero for μ–a.e. x:*

$$\lim_{r\downarrow 0}\frac{\mu(B_r(x))}{r^\alpha}\,.$$

Then either $\mu=0$, or α is a natural number $k\leq n$. In the latter case, a measure μ satisfies the requirement above if and only if there exists a Borel measurable function f and a countable collection $\{\Gamma_i\}$ of Lipschitz k–dimensional submanifolds of $\mathbb{R}^n$ such that

$$\mu(A)\;=\;\sum_i\int_{\Gamma_i\cap A}f(x)\,d\mathrm{Vol}^k(x)\qquad \textit{for any Borel set } A.$$

Here Vol^k denotes the natural k–dimensional volume measure that a Lipschitz submanifold inherits as a subset of $\mathbb{R}^n$.

The first part of Theorem 1.1, (i.e. if μ is nontrivial then α must be integer) was proved by Marstrand in [**17**]. The second part is trivial when $k=0$ and $k=n$. The first nontrivial case, $k=1$ and $n=2$, was proved by Besicovitch in his pioneering work [**2**], though in a different framework (Besicovitch's statement dealt with sets instead of measures). Besicovitch's theorem was recast in the framework above in [**24**], and in [**23**] it was extended to the case $k=1$ and generic n. The higher dimensional version remained a long standing problem. Marstrand in [**16**] made a major contribution to its solution. His ideas were sufficient to prove a weaker theorem for 2–dimensional sets in $\mathbb{R}^3$, which was later generalized by Mattila in [**18**] to arbitrary dimensions and codimensions.

The problem was finally solved by Preiss in [**25**]. His proof starts from Marstrand's work but he introduces many new and interesting ideas. Although the excellent book of Mattila [**21**] gives a summary of this proof, many details and some important ideas were not documented. As far as I know, the only reference for the proof of the second part of Theorem 1.1 is Preiss' paper itself.

As a measure of the complexity of the subject, we remark that natural generalizations of Marstrand, Mattila, and Preiss' theorems proved to be quite hard; see for instance [**12**] and [**13**].

Actually, in [**25**] Preiss proved the following stronger quantitative version of the second part of Theorem 1.1:

Theorem 1.2. *For any pair of nonnegative integers $k \leq n$ there exists a constant $c(k,n) > 1$ such that the following holds. If μ is a locally finite measure on $\mathbb{R}^n$ and*

$$0 \;<\; \limsup_{r\downarrow 0} \frac{\mu(B_r(x))}{r^k} \;<\; c(k,n) \liminf_{r\downarrow 0} \frac{\mu(B_r(x))}{r^k} \;<\; \infty \qquad \text{for } \mu\text{–a.e. } x \in \mathbb{R}^n,$$

then the same conclusion as for Theorem 1.1 holds.

The proof of this statement is longer and more difficult. On the other hand, most of the deep ideas contained in [**25**] are already needed to prove Theorem 1.1. Therefore, I decided to focus on Theorem 1.1.

Despite the depth of Theorem 1.1, no substantial knowledge of geometric measure theory is needed to read these notes. Indeed, the only prerequisites are:

- Some elementary measure theory;
- Some classical covering theorems and the Besicovitch Differentiation Theorem;
- Rademacher's Theorem on the almost everywhere differentiability of Lipschitz maps;
- The definition of Hausdorff measures and a few of their elementary properties.

All the fundamental definitions, propositions, and theorems are given in Chapter 2, together with references on where to find them.

The reader will note that I do not assume any knowledge of rectifiable sets. I define them in Chapter 4, where I prove some of their basic properties. The material of Chapter 4 can be found in other books and Mattila's book is a particularly good reference for Chapter 3 and Chapter 5. However, there are two good reasons for including Chapters 3, 4, and 5 in these notes:

(a) To make these notes accessible to people who are not experts in the field;
(b) To show the precursors of some ideas of Preiss' proof, in the hope that it makes them easier to understand.

These two reasons have also been the main guidelines in presenting the proofs of the various propositions and theorems. Therefore, some of the proofs are neither the shortest nor the most elegant available in the literature. For instance, as far as I know, the shortest and most elegant proof of Marstrand's Theorem (see Theorem 3.1) uses a beautiful result of Kirchheim and Preiss (see Theorem 3.11 in [**10**]). However, I have chosen to give Marstrand's original proof because the "moments" introduced by Preiss (which play a major role in his proof; see Chapters 7, 8, and 9) are reminiscent of the "barycenter" introduced by Marstrand (see (3.17)).

Similarly, I have not hesitated to sacrifice generality, whenever this seemed to make the statements, the notation, or the ideas more transparent. Therefore, many other remarkable facts proved by Preiss in [**25**] are not mentioned in these notes.

As already mentioned above, Chapter 2 is mostly a list of prerequisites on measure theory. In Chapter 3, we prove the classical result of Marstrand that if $\alpha \in \mathbb{R}$ and $\mu \neq 0$ satisfy the assumption of Theorem 1.1, then α is an integer. In this chapter we also introduce the notion of tangent measure.

In Chapter 4 we define rectifiable sets and rectifiable measures and we prove the Area Formula and a classical rectifiability criterion. As an application of these tools we give a first characterization of rectifiable measures in terms of their tangent measures.

In Chapter 5 we prove a deeper rectifiability criterion, due to Marstrand for 2–dimensional sets in $\mathbb{R}^3$ and extended by Mattila to general dimension and codimension. This rectifiability criterion plays a crucial role in the proof of Theorem 1.1.

In Chapter 6 we give an overview of Preiss' proof of Theorem 1.1. In this chapter we motivate some of its difficulties and we split the proof into three main steps, each of which is taken in one of the subsequent three chapters. Chapter 10 is a collection of open problems connected to the the various topics of the notes, which I collected together with Bernd Kichheim.

In Appendix A we prove the Kirchheim–Preiss Theorem on the analiticity of the support of uniformly distributed euclidean measures, whereas Appendix B contains some useful elementary computations on Gaussian integrals.

I wish to thank Matteo Focardi and Andrew Lorent, who carefully read these notes and helped me with many comments and suggestions, and Filippo Pellandini, David Pumberger, and Stefan Wenger, who followed the last part of my course.

CHAPTER 2

Notation and preliminaries

In this section, we gather some basic facts which will be used later in these notes. For a proof of the various theorems and propositions listed in the next sections, the reader is referred to Chapter 1 and Sections 2.3, 2.4, 2.5, and 2.8 of [**1**].

1. General notation and measures

The topological closure of a set U and its topological boundary will be denoted respectively by $\overline{U}$ and ∂U. Given $x \in \mathbb{R}^n$ and $r > 0$, we will use $B_r(x)$, $\overline{B}_r(x)$, and $\partial B_r(x)$ to denote, respectively, the open ball centered at x of radius r, its closure, and its boundary. A k–dimensional linear subspace of $\mathbb{R}^m$ will be called a k–dimensional linear plane. When V is a k–dimensional linear plane and $x \in \mathbb{R}^m$, the set $x + V$ will be called a *k–dimensional affine plane*. We will simply use the word "plane" when there is no ambiguity as to whether we mean a linear or an affine plane. When x and y are vectors of $\mathbb{R}^n$, we will denote by $\langle x, y\rangle$ their scalar product. When A and B are matrices and x is a vector, we will denote by $A \cdot B$ and $A \cdot x$ the usual product of matrices and the usual product of a matrix and a column vector.

In these notes we will always consider nonnegative measures μ, though many theorems can be generalized to real and vector–valued measures with almost no effort. μ–measurable sets and μ–measurable functions are defined in the usual way. The Lebesgue measure on $\mathbb{R}^n$ is denoted by $\mathscr{L}^n$.

When $E \subset U$ and μ is a measure on U, we will denote by $\mu \llcorner E$ the measure defined by

$$[\mu \llcorner E]\,(A) \ := \ \mu(A \cap E)\,.$$

If f is a nonnegative μ–measurable function, then we denote by $f\mu$ the measure defined by

$$[f\mu](A) \ := \ \int_A f\,d\mu\,.$$

We say that a measure μ is Borel regular if the Borel sets are μ–measurable and if for every μ–measurable set A there exists a Borel set B such that $A \subset B$ and $\mu(B \setminus A) = 0$. We say that a Borel measure μ is locally finite if $\mu(K) < \infty$ for every compact set K. All the measures considered in these notes are Borel regular and, except for the Hausdorff measures (see below), they are all locally finite. Moreover, even when dealing with the Hausdorff measure $\mathscr{H}^k$, we will always work with its restrictionS to Borel sets E with locally finite $\mathscr{H}^k$ measure, i.e. such that $\mathscr{H}^k(E \cap K) < \infty$ for every compact set K. Hence, in practice, we will always deal with measures which are Borel regular and locally finite. For these measures, the following proposition holds true (see Proposition 1.43 of [**1**]).

PROPOSITION 2.1. *Let μ be a Borel regular and locally finite measure on $\mathbb{R}^n$. If E is a Borel set such that $\mu(E) < \infty$, then for every $\varepsilon > 0$ there exists a compact set K and an open set U, such that $K \subset E \subset U$ and $\mu(U \setminus K) < \varepsilon$.*

Sometimes, when comparing two different measures μ and ν on an open set A we will use the *total variation* of $\mu - \nu$ on A, which is denoted by $|\mu - \nu|(A)$ and is defined as

$$|\mu - \nu|(A) \; := \sup_{\varphi \in C_c(A), |\varphi| \leq 1} \int \varphi \, d(\mu - \nu) \, .$$

We will say that the μ–measurable function f is *Lebesgue continuous at a point x with respect to the measure μ* if we have

$$\lim_{r \downarrow 0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} \big| f(y) - f(x) \big| \, dy \; = \; 0 \, .$$

When μ is the Lebesgue measure, we will simply say that f is Lebesgue continuous at x. The following is an application of the Besicovitch Differentiation Theorem 2.10 (compare with Corollary 2.23 of [**1**]).

PROPOSITION 2.2. *If μ is a locally finite measure and $f \in L^1(\mu)$, then for μ–a.e. x, f is Lebesgue continuous at x with respect to μ.*

2. Weak* convergence of measures

As usual, we endow the space $C_c(\mathbb{R}^n)$ of continuous compactly supported functions with the topology of uniform convergence on compact sets. This means that $\varphi_j \to \varphi$ if

- there exists a compact set K such that $\mathrm{supp}\,(\varphi_j) \subset K$ for every n;
- $\varphi_j \to \varphi$ uniformly.

If μ is a locally finite measure on $\mathbb{R}^n$, then the map

$$\varphi \;\to\; \int \varphi d\mu$$

induces a continuous linear functional on $C_c(\mathbb{R}^n)$. The converse is also true:

THEOREM 2.3 (Riesz' Representation Theorem). *Let $L : C_c(\mathbb{R}^n) \to \mathbb{R}$ be a linear functional such that $L(\varphi) \geq 0$ for every $\varphi \geq 0$. Then there exists a locally finite nonnegative measure μ such that*

$$L(\mu) \; := \; \int \varphi \, d\mu \, .$$

Therefore, it is natural to endow the space of locally finite Euclidean measures $\mathcal{M}$ with the topology of the dual space of $C_c(\mathbb{R}^n)$:

DEFINITION 2.4. *Let $\{\mu_j\}$ be a sequence of locally finite nonnegative measures on $\mathbb{R}^n$. We say that μ_j converges weakly* to μ (and we write $\mu_j \overset{*}{\rightharpoonup} \mu$) if*

$$\lim_{j \uparrow \infty} \int \varphi \, d\mu_j \; = \; \int \varphi \, d\mu$$

for every $\varphi \in C_c(\mathbb{R}^n)$.

We will often use the fact that if for every bounded open set A we have $|\mu_j - \mu|(A) \to 0$, then $\mu_j \overset{*}{\rightharpoonup} \mu$.

Note that if $\mu_j \overset{*}{\rightharpoonup} \mu$, then $\{\mu_j\}$ is uniformly locally bounded, that is, for every compact set K there exists a constant C_K such that $\mu_j(K) \le C_K$ for every $j \in \mathbb{N}$. Moreover, since $\mathcal{M}$ is the dual of the topological vector space $C_c(\mathbb{R}^n)$, the weak* topology defined above enjoys the following compactness property:

Proposition 2.5. *Let $\{\mu_j\}$ be a sequence of uniformly locally bounded measures. Then there exists a subsequence $\{\mu_{j_i}\}$ and a locally finite measure μ such that $\mu_{j_i} \overset{*}{\rightharpoonup} \mu$.*

Moreover, since the topological vector space $C_c(\mathbb{R}^n)$ is separable, the following metrizability property is well known.

Proposition 2.6 (Metrizability of weak* convergence). *Let $\mathcal{M}(\mathbb{R}^n)$ be the set of non-negative locally finite measures. Then there exists a distance d on $\mathcal{M}(\mathbb{R}^n)$ such that*

$$\mu_j \overset{*}{\rightharpoonup} \mu \qquad \text{if and only if} \qquad d(\mu_j,\mu) \to 0 \text{ and } \{\mu_j\} \text{ is locally uniformly bounded.}$$

For the reader's convenience we include a proof of this proposition.

Proof. Let $G := \{f_i\} \subset C_c(\mathbb{R}^n)$ be a countable dense set. That is, for every $f \in C_c(\mathbb{R}^n)$ there exists a sequence $\{f_{i(j)}\} \subset G$ such that $f_{i(j)} \to f$ and the supports of $f_{i(j)}$ are all contained in a compact set K_f.

For $i \in \mathbb{N}$ and $\mu, \nu \in \mathcal{M}$ we define

$$d_i(\mu,\nu) \;:=\; \left|\int f_i\, d\mu - \int f_i\, d\nu\right| .$$

Then we set

$$d(\mu,\nu) \;:=\; \sum_{i=1}^{\infty} 2^{-i} \min\left\{d_i(\mu,\nu)\,,\,1\right\} .$$

Clearly d defines a distance. Indeed, if $d(\mu,\nu)=0$ then $\int f d\mu = \int f d\nu$ for every $f \in C_c(\mathbb{R}^n)$, which implies $\mu = \nu$. Hence, it suffices to check the triangle inequality, which follows easily from

$$d_i(\mu,\zeta) \;\le\; d_i(\mu,\nu) + d_i(\nu,\zeta) .$$

Now assume $\mu_j \overset{*}{\rightharpoonup} \mu$. Then for each $f_i \in G$ we have

$$\lim_{j\to\infty} \int f_i\, d\mu_j \;=\; \int f_i\, d\mu . \tag{2.1}$$

After fixing $1 > \delta > 0$ we select $N_0 > 0$ such that $\sum_{i>N_0} 2^{-i} < \delta/2$. From (2.1) we conclude that there exists an $N_1 > 0$ such that

$$d_i(\mu_j,\mu) \;=\; \left|\int f_i d\mu - \int f_i d\mu_j\right| \;\le\; \frac{\delta}{2N_0} \qquad \text{for every } i \le N_0 \text{ and } j \ge N_1 . \tag{2.2}$$

Therefore, for $j \ge N_1$ we have

$$d(\mu_j,\mu) \;\le\; \sum_{i=1}^{N_0} 2^{-1} d_i(\mu_j,\mu) + \sum_{i>N_0} 2^{-i} \;\le\; \frac{\delta}{2} + \frac{\delta}{2} \;=\; \delta .$$

We conclude that $d(\mu_j, \mu) \to 0$.

On the other hand, assume that $d(\mu_j, \mu) \to 0$ and that $\{\mu_j\}$ is locally uniformly bounded. Let $\varphi \in C_c(\mathbb{R}^n)$. From our assumptions there exists a compact set K which contains $\operatorname{supp}(\varphi)$ and a sequence $\{f_i\} \subset G$ such that $f_i \to f$ uniformly and $\operatorname{supp}(f_i) \subset K$.

Let M be such that $\mu(K) + \mu_j(K) \leq M$ for every j. For any given $\varepsilon > 0$ we can choose f_i in the sequence above such that $\|\varphi - f_i\|_\infty \leq \varepsilon/(2M)$. Now, since $d_i(\mu_j, \mu) \to 0$, we can choose N such that

$$d_i(\mu_j, \mu) \;\leq\; \frac{\varepsilon}{2} \qquad \text{for every } j \geq N.$$

Therefore, we can compute

$$\begin{aligned}\left|\int \varphi d\mu_j - \int \varphi d\mu\right| \;&\leq\; \left|\int f_i d\mu_j - \int f_i d\mu\right| + \left|\int (f_i - \varphi) d\mu_j\right| + \left|\int (f_i - \varphi) d\mu\right| \\ &\leq\; \frac{\varepsilon}{2} + \|\varphi - f_i\|_\infty \big(\mu(K) + \mu_j(K)\big) \;\leq\; \varepsilon\,.\end{aligned}$$

Therefore, we conclude that

$$\lim_{j\uparrow\infty} \int \varphi\, d\mu_j \;=\; \int \varphi\, d\mu\,.$$

The arbitrariness of φ implies that $\mu_j \overset{*}{\rightharpoonup} \mu$. □

Finally we conclude this section with a technical proposition which will be very useful in many situations.

Proposition 2.7. *Let ν_i be a sequence of measures such that $\nu_i \overset{*}{\rightharpoonup} \nu$. Then*

- $\liminf_i \nu_i(A) \geq \nu(A)$ *for every open set A;*
- $\limsup_i \nu_i(K) \leq \nu(K)$ *for every compact set K.*

Therefore,

- $\nu_i(A) \to \nu(A)$ *for every bounded open set A such that $\nu(\partial A) = 0$;*
- *For any point x there exists a set $S_x \subset \mathbb{R}^+$ at most countable such that*

$$\nu_i(B_\rho(x)) \;\to\; \nu(B_\rho(x)) \qquad \textit{for every } \rho \in \mathbb{R}^+ \setminus S_x.$$

Proof. Let ν_i and ν be as in the statement of the proposition and assume A is open. Let $\{\varphi_j\} \subset C_c(A)$ be such that $0 \leq \varphi_j \leq 1$ and $\varphi_j(x) \to 1$ for every $x \in A$. Since $\nu_i(A) \geq \int \varphi_j d\nu_i$ for every j and i, we have

$$\liminf_{i\uparrow\infty} \nu_i(A) \;\geq\; \liminf_{i\uparrow\infty} \int \varphi_j\, d\nu_i \;=\; \int \varphi_j\, d\nu \qquad \text{for every } j.$$

Letting $j \uparrow \infty$ we obtain

$$\liminf_{i\uparrow\infty} \nu_i(A) \;\geq\; \nu(A)\,. \tag{2.3}$$

Consider now K compact and fix $\varepsilon > 0$. Let U be an open set such that $K \subset U$ and $\nu(U \setminus K) < \varepsilon$. Now, fix $\varphi \in C_c(U)$ such that $0 \leq \varphi \leq 1$ and $\varphi = 1$ on K. Then we have

$$\limsup_{i\uparrow\infty} \nu_i(K) \;\leq\; \limsup_{i\uparrow\infty} \int \varphi\, d\nu_i \;=\; \int \varphi\, d\nu \;\leq\; \nu(U) \;< \nu(K) + \varepsilon\,.$$

The arbitrariness of ε gives

$$\limsup_{n\uparrow\infty} \nu_i(K) \;\leq\; \nu(K)\,. \tag{2.4}$$

Next let A be a bounded open set such that $\nu(\partial A) = 0$. Then, $\overline{A}$ is compact and, by (2.3) and (2.4),

$$\liminf_i \nu_i(A) \;\geq\; \nu(A) \;=\; \nu(\bar{A}) \;\geq\; \limsup_i \nu(\bar{A}) \;\geq\; \limsup_i \nu_i(A)\,.$$

Finally, given x, we consider the set

$$S_x \;:=\; \left\{r \in \mathbb{R}^+ \;:\; \nu(\partial B_r(x)) > 0\right\}\,.$$

According to what we have proved so far, we have

$$\lim_{i\uparrow\infty} \nu_i(B_r(x)) \;=\; \nu(B_r(x)) \qquad \text{for any } r \in \mathbb{R}^+ \setminus S_x.$$

Since ν is locally finite, S_x is at most countable. □

3. Covering theorems and differentiation of measures

In these notes we will use two well–known covering theorems. For the first, we refer the reader to Theorem 2.1 of [**21**], and for the second, to Theorem 2.19 of [**1**].

Theorem 2.8 ($5r$–Covering Theorem). *Let $\mathcal{B}$ be a family of balls of the Euclidean space $\mathbb{R}^n$ such that the supremum of their radii is finite. Then there exists a countable subset $\mathcal{C} = \{B_{r_i}(x_i)\}_{i\in\mathbb{N}}$ of $\mathcal{B}$ such that:*

- *The balls $B_{r_i}(x_i)$ are pairwise disjoint;*
- $\bigcup_{B\in\mathcal{B}} B \;\subset\; \bigcup_{i\in\mathbb{N}} B_{5r_i}(x_i)$.

Theorem 2.9 (Besicovitch–Vitali Covering Theorem). *Let A be a bounded Borel Euclidean set and $\mathcal{B}$ a collection of* closed *balls such that for every $x \in A$ and every $r > 0$ there exists a ball $\overline{B}_\rho(x) \in \mathcal{B}$ with radius $\rho < r$. If μ is a locally finite measure, then there exists a countable subset $\mathcal{C} \subset \mathcal{B}$ of pairwise disjoint balls such that $\mu\left(A \setminus \bigcup_{\overline{B}\in\mathcal{C}} \overline{B}\right) = 0$.*

The Besicovitch–Vitali Covering Theorem is the main tool for proving the following differentiation theorem for measures (see Theorem 2.22 of [**1**]):

Theorem 2.10 (Besicovitch Differentiation of Measures). *Let μ and ν be locally finite Euclidean measures. Then the limit*

$$f(x) \;:=\; \lim_{r\downarrow 0} \frac{\nu(B_r(x))}{\mu(B_r(x))}$$

exists at μ–a.e. point $x \in \operatorname{supp}(\mu)$. Moreover, the Radon–Nikodym decomposition of ν with respect to μ is given by $f\mu + \nu \llcorner E$, where

$$E \;:=\; \big(\mathbb{R}^n \setminus \operatorname{supp}(\mu)\big) \cup \left\{x \in \operatorname{supp}(\mu) \;:\; \lim_{r\downarrow 0} \frac{\nu(B_r(x))}{\mu(B_r(x))} = \infty\right\}\,.$$

4. Hausdorff measures

For any nonnegative real number α we define the constant ω_α to be $\pi^{\alpha/2}\Gamma(1+\alpha/2)$, where

$$\Gamma(t) \;:=\; \int_0^\infty s^{t-1} e^{-s}\, ds\,.$$

When α is an integer, ω_α is equal to the $\mathscr{L}^\alpha$ measure of the Euclidean unit ball of $\mathbb{R}^\alpha$ (see Proposition B.1).

We define the α–dimensional Hausdorff measure on $\mathbb{R}^n$ in the usual way (cf. Definition 2.46 of [**1**]):

Definition 2.11. *Let $E \subset \mathbb{R}^n$. The α–dimensional Hausdorff measure of E is denoted by $\mathscr{H}^\alpha(E)$ and defined by*

$$\mathscr{H}^\alpha(E) := \lim_{\delta\downarrow 0} \mathscr{H}^\alpha_\delta(E)$$

where $\mathscr{H}^\alpha_\delta(E)$ is defined as

$$\mathscr{H}^\alpha_\delta(E) := \frac{\omega_\alpha}{2^\alpha} \inf\left\{ \sum_{i\in I} \big(\mathrm{diam}\,(E_i)\big)^\alpha \,\Big|\, \mathrm{diam}\,(E_i) < \delta\,,\ E \subset \bigcup_{i\in I} E_i \right\}.$$

In the following proposition we summarize some important properties of the Hausdorff measure (see Propositions 2.49 and 2.53 of [**1**]).

Proposition 2.12.

(i) *The measures $\mathscr{H}^\alpha$ are Borel.*
(ii) *They are translation–invariant and $\mathscr{H}^\alpha(\lambda E) = \lambda^\alpha \mathscr{H}^\alpha(E)$ for every positive λ.*
(iii) *If $\alpha > \alpha' > 0$ then $\mathscr{H}^\alpha(E) > 0 \Longrightarrow \mathscr{H}^{\alpha'}(E) = \infty$.*
(iv) *If $f : \mathbb{R}^m \to \mathbb{R}^n$ is a Lipschitz map, then $\mathscr{H}^\alpha(f(E)) \leq (\mathrm{Lip}\,(f))^\alpha \mathscr{H}^\alpha(E)$.*
(v) *The n–dimensional Hausdorff measure on $\mathbb{R}^n$ coincides with the Lebesgue measure.*

Point (iii) allows the *Hausdorff dimension* of a set E to be defined as the infimum of the α's such that $\mathscr{H}^\alpha(E) = 0$. Proposition 2.13 below is a direct consequence of (v). Before stating the proposition, we first need to introduce the definition of the push–forward of a measure. If μ is a measure on $\mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}^n$ is μ–measurable, then we define the measure $f_\# \mu$ as

$$\left[f_\#\mu\right](A) := \mu\big(f^{-1}(A)\big)\,.$$

Proposition 2.13. *Let $V \subset \mathbb{R}^n$ be a k–dimensional affine plane. Fix a system of orthonormal coordinates $x_1, \ldots, x_k, y_1, \ldots, y_{n-k}$ such that $V = \{y_1 = \ldots = y_{n-k} = 0\}$. Denote by $\iota : \mathbb{R}^k \to \mathbb{R}^n$ the map $x \to (x, 0)$. Then $\mathscr{H}^k \llcorner V = \iota_\# \mathscr{L}^k$.*

We end this section by defining the α–densities of Euclidean measures and sets at a given point x (cf. Definition 2.55 of [**1**]).

Definition 2.14. *Let μ be a locally finite Euclidean measure and α a nonnegative number. Then we define the* upper *(resp.* lower*) α–*density *of μ at x as*

$$\theta^{*\alpha}(\mu, x) := \limsup_{r\downarrow 0} \frac{\mu(B_r(x))}{\omega_\alpha r^\alpha} \qquad \theta^\alpha_*(\mu, x) := \liminf_{r\downarrow 0} \frac{\mu(B_r(x))}{\omega_\alpha r^\alpha}\,.$$

*When the two quantities coincide, we simply speak of the α–*density *of μ at x, denoted by $\theta^\alpha(\mu, x)$.*

If E is a Borel set, we define the α–densities of E at x as

$$\begin{aligned} \theta^{*\alpha}(E, x) &:= \theta^{*\alpha}\big(\mathscr{H}^\alpha \llcorner E, x\big) \\ \theta^\alpha_*(E, x) &:= \theta^\alpha_*\big(\mathscr{H}^\alpha \llcorner E, x\big) \\ \theta^\alpha(E, x) &:= \theta^\alpha\big(\mathscr{H}^\alpha \llcorner E, x\big)\,. \end{aligned}$$

Concerning the relations between densities and measures, we have two useful propositions which both follow from Proposition 2.56 of [**1**].

Proposition 2.15. *Let E be a Borel set and α a nonnegative number such that $\mathscr{H}^\alpha(E) < \infty$. Then*

- $\theta^{*\alpha}(E,x) = 0$ *for $\mathscr{H}^\alpha$–a.e. $x \in \mathbb{R}^n \setminus E$;*
- $2^{-\alpha} \leq \theta^{*\alpha}(E,x) \leq 1$ *for $\mathscr{H}^\alpha$–a.e. $x \in E$.*

Proposition 2.16. *Let μ be a measure and α a nonnegative real number such that*

$$0 \ < \ \theta^{\alpha *}(\mu, x) \ < \ \infty \qquad \text{for } \mu\text{–a.e. } x.$$

Then there exists an α–dimensional set E and a Borel function f such that $\mu = f\mathscr{H}^\alpha \llcorner E$.

5. Lipschitz functions

Let E be a subset of $\mathbb{R}^n$. $f : E \to \mathbb{R}^k$ is a Lipschitz function if there exists a constant K such that

$$|f(x) - f(y)| \ \leq \ K|x-y| \qquad \forall x, y \in E\,. \tag{2.5}$$

The smallest number K for which inequality (2.5) holds is called the *Lipschitz constant* of f and we denote it by $\mathrm{Lip}\,(f)$.

The following Proposition has a very elementary proof:

Proposition 2.17. *Let $f : \mathbb{R}^k \supset G \to \mathbb{R}^m$ be a Lipschitz function. Then there exists a Lipschitz function $\tilde{f} : \mathbb{R}^k \to \mathbb{R}^m$ such that $\tilde{f}|_G = f$.*

Proof. If $m = 1$ we set

$$\tilde{f}(x) \ := \ \inf_{y \in E} f(y) + \mathrm{Lip}\,(f)|y-x|\,. \tag{2.6}$$

It is easy to check that $\tilde{f}$ is Lipschitz and is an extension of f. When $m > 1$ we use (2.6) to extend each component of the vector f. □

Remark 2.18. *Note that for $m = 1$ the function $\tilde{f}$ defined in* (2.6) *satisfies* $\mathrm{Lip}\,(\tilde{f}) = \mathrm{Lip}\,(f)$. *For $m > 1$ the extension suggested above does not have this property in general. However, there does exist an extension $\hat{f}$ such that* $\mathrm{Lip}\,(\hat{f}) = \mathrm{Lip}\,(f)$. *This statement is called Kirszbraun's Theorem, and it is considerably more difficult to prove (see 2.10.43 of* [**8**]*).*

The following are two remarkable theorems concerning Lipschitz functions. In these notes we will use only the first, but we include the second because it often gives very good insight into the various properties of Lipschitz functions. For a proof of Theorem 2.19, see Theorem 2.14 of [**1**]. For a proof of Theorem 2.20, see Theorem 3.1.16 of [**8**].

Theorem 2.19 (Rademacher). *Let $f : \mathbb{R}^n \supset E \to \mathbb{R}^k$ be a Lipschitz function. Then f is differentiable at $\mathscr{L}^n$–a.e. $x \in E$, that is, for $\mathscr{L}^n$–a.e. $x \in E$ there exists a linear map $df_x : \mathbb{R}^n \to \mathbb{R}^k$ such that*

$$\lim_{y \in E\,,\, y \to x} \frac{|f(y) - f(x) - df_x(y-x)|}{|y-x|} \ = \ 0\,.$$

Theorem 2.20 (Whitney's extension theorem). *Let $f : \mathbb{R}^n \supset E \to \mathbb{R}^k$ be a Lipschitz function. For every $\varepsilon > 0$ there exists a function $\tilde{f} \in C^1(\mathbb{R}^n, \mathbb{R}^k)$ such that $\mathscr{L}^n(\{x \in E : f(x) \neq \tilde{f}(x)\}) < \varepsilon$.*

6. The Stone–Weierstrass Theorem

In some approximation arguments we will make use of the classical Stone–Weierstrass Theorem (see Theorem 7.31 of [**27**]):

DEFINITION 2.21. *Let $\mathcal{F}$ be a family of real functions on the set E. Then we say that*

- $\mathcal{F}$ separates the points *if for every $x \neq y \in E$ there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$;*
- $\mathcal{F}$ vanishes at no point of E *if for every $x \in E$ there exists $f \in \mathcal{F}$ such that $f(x) \neq 0$.*

THEOREM 2.22 (Stone–Weierstrass). *Let K be a compact set and $\mathcal{A} \subset C(K)$ be an algebra of functions which separates the points and vanishes at no point. Then for every $f \in C(K)$ there exists $\{f_j\} \subset \mathcal{A}$ such that $f_j \to f$ uniformly.*

CHAPTER 3

Marstrand's Theorem and tangent measures

The goal of this chapter is to prove the following beautiful result of Marstrand:

THEOREM 3.1 (Marstrand's Theorem). *Let μ be a measure on $\mathbb{R}^n$, α a nonnegative real number, and E a Borel set with $\mu(E) > 0$. Assume that*

$$0 \;<\; \theta^{\alpha}_{*}(\mu, x) = \theta^{\alpha *}(\mu, x) < \infty \qquad \text{for } \mu\text{–a.e. } x \in E. \tag{3.1}$$

Then α is an integer.

This theorem was first proved in [**17**]. Actually, in [**17**], the author proved a much stronger result, which provides important information on the measures μ satisfying (3.1) for α integer. This second part of Marstrand's result is stated in Remark 3.10 and will be proved in Chapter 6 (cp. with Theorem 6.8).

Our presentation is very close to that of chapter 14 of [**21**], particularly in that we will use tangent measures.

Blow up. The first idea of the proof is that if for some α there exists a nontrivial μ which satisfies (3.1), then, via a "blow–up" procedure, we can produce a second (nontrivial) measure ν which satisfies a much stronger condition than (3.1). In particular, ν will be an α–uniform measure in the following sense:

DEFINITION 3.2 (α–uniform measures). *We say that a measure μ is α–uniform if*

$$\mu(B_r(x)) \;=\; \omega_\alpha r^\alpha \qquad \text{for every } x \in \operatorname{supp}(\mu) \text{ and every } r > 0.$$

We denote by $\mathcal{U}^\alpha(\mathbb{R}^n)$ the set of α–uniform measures ν such that $0 \in \operatorname{supp}(\nu)$.

This particular choice of the constant ω_α will be convenient later since it ensures $\mathscr{H}^k \llcorner V \in \mathcal{U}^k(\mathbb{R}^n)$ for every k–dimensional linear plane $V \subset \mathbb{R}^n$. We warn the reader that there exist k–uniform measures which are not of the form $\mathscr{H}^k \llcorner V$: An example of such a measure is given in Section 1 of Chapter 6. This striking fact will play a crucial role in the last part of these notes (see the introduction to Chapter 6).

The "blow–up" procedure is better described after introducing the notion of tangent measure. Not only will this notion simplify the discussion of this chapter, but it will also be extremely useful in later chapters.

DEFINITION 3.3 (Tangent measures). *Let μ be a measure, $x \in \mathbb{R}^n$, and r be a positive real number. Then the measure $\mu_{x,r}$ is defined by*

$$\mu_{x,r}(A) \;=\; \mu(x + rA) \qquad \text{for all Borel sets } A \subset \mathbb{R}^n.$$

For any nonnegative real number α, we denote by $\operatorname{Tan}_\alpha(\mu, x)$ the set of all measures ν for which there exists a sequence $r_i \downarrow 0$ such that

$$\frac{\mu_{x,r_i}}{r_i^\alpha} \overset{*}{\rightharpoonup} \nu \qquad \text{in the sense of measures.}$$

$\mathrm{Tan}_\alpha(\mu, x)$ is a subset of $\mathrm{Tan}(\mu, x)$, the set of *tangent measures to μ at x*, first introduced by Preiss in [**25**]. In his definition, Preiss considers all measures ν which are weak limits of $c_i\mu_{x,r_i}$ for some choice of a vanishing sequence $\{r_i\}$ and of a positive sequence $\{c_i\}$. However, in all cases considered in these notes, $\mathrm{Tan}_\alpha(\mu, x)$ contains all the information about $\mathrm{Tan}(\mu, x)$.

Using the language of tangent measures, the first ingredient of the proof of Theorem 3.1 is given by the following proposition, which roughly says that at almost every point x, at sufficiently small scale, μ is close to a nontrivial α–uniform measure. Nowadays, arguments like that of Proposition 3.4 are considered to be quite standard in Geometric Measure Theory.

PROPOSITION 3.4. *Let μ be as in Theorem 3.1, then for μ–a.e. $x \in E$ we have*

$$\emptyset \neq \mathrm{Tan}_\alpha(\mu, x) \subset \left\{\theta^\alpha(\mu, x)\nu : \nu \in \mathcal{U}^\alpha(\mathbb{R}^n)\right\}.$$

α–**Uniform measures.** The second step in the proof of Theorem 3.1 is to show that the following proposition is valid.

PROPOSITION 3.5. *If $\mathcal{U}^\alpha(\mathbb{R}^n) \neq \emptyset$, then α is a nonnegative integer less than or equal to n.*

The proof of this Proposition is the core of this chapter. Here we briefly describe the scheme of Marstrand's approach.

SKETCH OF THE PROOF OF PROPOSITION 3.5.

(a) The Besicovitch Differentiation Theorem gives $\mathcal{U}^\alpha(\mathbb{R}^k) = \emptyset$ for every $\alpha > k$ (see Remark 3.14).

(b) We will show that, if $\alpha < k$, then

$$\mathcal{U}^\alpha(\mathbb{R}^k) \neq \emptyset \quad \Longrightarrow \quad \mathcal{U}^\alpha(\mathbb{R}^{k-1}) \neq \emptyset. \tag{3.2}$$

(c) Arguing by contradiction, assume that $\mathcal{U}^\alpha(\mathbb{R}^n) \neq \emptyset$ for some $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$. Let $k := [\alpha] < \alpha < n$ and iterate $n - [\alpha]$ times (3.2). We conclude that $\mathcal{U}^\alpha(\mathbb{R}^k) \neq \emptyset$, which contradicts (a).

□

Clearly, the key point of this scheme is (b). Its proof relies again on a "blow–up" procedure, which we split into the following lemmas. The first is a trivial remark:

LEMMA 3.6. *Let $\alpha \geq 0$, $\mu \in \mathcal{U}^\alpha(\mathbb{R}^k)$, and $x \in \mathrm{supp}\,(\mu)$. Then $\emptyset \neq \mathrm{Tan}_\alpha(\mu, x) \subset \mathcal{U}^\alpha(\mathbb{R}^k)$.*

The second is an elementary geometric observation (see Section 2 and Figure 1).

LEMMA 3.7. *Let $0 \leq \alpha < k$ and $\mu \in \mathcal{U}^\alpha(\mathbb{R}^k)$. Then there exists $y \in \mathrm{supp}\,(\mu)$ and a system of coordinates $x_1, \ldots, x_k$ on $\mathbb{R}^k$ such that*

$$\mathrm{supp}\,(\nu) \subset \{x_1 \geq 0\} \qquad \textit{for every } \nu \in \mathrm{Tan}_\alpha(\mu, y). \tag{3.3}$$

The last is the core of Marstrand's proof:

LEMMA 3.8. *Let $0 \leq \alpha < k$ and $\nu \in \mathcal{U}^\alpha(\mathbb{R}^k)$. If $\mathrm{supp}\,(\nu) \subset \{x_1 \geq 0\}$, then*

$$\mathrm{supp}\,(\tilde{\nu}) \subset \{x_1 = 0\} \qquad \textit{for every } \tilde{\nu} \in \mathrm{Tan}_\alpha(\nu, 0). \tag{3.4}$$

From these three Lemmas we easily conclude that (b) holds using the following procedure:

- We fix $\mu \in \mathcal{U}^\alpha(\mathbb{R}^n)$ and we apply Lemma 3.7 in order to find a $y \in \operatorname{supp}(\mu)$ that satisfies (3.3).
- Consider $\nu \in \operatorname{Tan}_\alpha(\mu, x)$. Then by Lemma 3.6 we have $\nu \in \mathcal{U}^\alpha(\mathbb{R}^n)$ and from (3.3) we obtain $\operatorname{supp}(\nu) \subset \{x_1 \geq 0\}$.
- Finally consider $\tilde{\nu} \in \operatorname{Tan}^\alpha(\nu, 0)$. Such a measure belongs to $\mathcal{U}^\alpha(\mathbb{R}^n)$ (again by Lemma 3.6) and its support is contained in the hyperplane $\{x_1 = 0\}$.

Therefore, $\tilde{\nu}$ can be seen naturally as an element of $\mathcal{U}^\alpha(\mathbb{R}^{n-1})$.

m–**Uniform measures.** Note that none of the lemmas above needs the assumption $\alpha \in \mathbb{R} \setminus \mathbb{N}$, which indeed plays a role only in the final argument by contradiction contained in (c). Moreover, the Besicovitch Differentiation Theorem gives $\mathcal{U}^k(\mathbb{R}^k) = \{\mathscr{L}^k\}$. Therefore, from the procedure outlined above and a standard diagonal argument we obtain the following:

COROLLARY 3.9. *Let m be an integer and $\mu \in \mathcal{U}^m(\mathbb{R}^n)$. Then there exists an m–dimensional linear plane $V \subset \mathbb{R}^n$ and two sequences $\{x_i\}$ and $\{r_i\}$ such that*

$$\frac{\mu_{x_i,r_i}}{r_i^m} \stackrel{*}{\rightharpoonup} \mathscr{H}^m \llcorner V \qquad \textit{in the sense of measures.}$$

REMARK 3.10. *Actually, in* [**17**] *Marstrand proved a much stronger result, which, in the language of tangent measures, says that:*

- *If α is an integer and μ satisfies the assumptions of Theorem 3.1, then the following holds for μ–a.e. x:*

$$\textit{There exists an } \alpha\textit{–dimensional plane } V \textit{ such that } \theta^\alpha(\mu, x)\mathscr{H}^\alpha \llcorner V \in \operatorname{Tan}_\alpha(\mu, x). \tag{3.5}$$

This statement is proved in Chapter 6 (cp. with Theorem 6.8) and it is the starting point of Preiss' Theorem (see the introduction to Chapter 6).

The Kirchheim–Preiss Regularity Theorem. Both Proposition 3.4 and Corollary 3.9 can be proved in a more direct way by using the following remarkable Theorem of Kirchheim and Preiss; see [**10**].

THEOREM 3.11. *Let μ be a measure of $\mathbb{R}^n$ such that*

$$\mu(B_r(x)) \;=\; \mu(B_r(y)) \qquad \textit{for every } x, y \in \operatorname{supp}(\mu) \textit{ and every } r > 0. \tag{3.6}$$

Then the support of μ is a real analytic variety, i.e. there exists an analytic function $H : \mathbb{R}^n \to \mathbb{R}$ such that $\operatorname{supp}(\mu) = \{H = 0\}$.

In Appendix A we include a proof of Theorem 3.11, taken from [**10**]. Recall that, if we exlude $Z = \mathbb{R}^n$ (which corresponds to the trivial case $H \equiv 0$), any analytic variety $Z \subset \mathbb{R}^n$ has a natural stratification

$$Z \;=\; \bigcup_{i=0}^{n-1} Z_i \,, \tag{3.7}$$

where each Z_i is an i–dimensional (possibly empty) analytic submanifold of $\mathbb{R}^n$. If μ satisfies (3.6) and Z is the analytic variety given by Theorem 3.11, then let k be largest i for which Z_i in (3.7) is not empty. Then Z is a rectifiable k–dimensional set and it is not difficult to show that $\mu = c\mathscr{H}^k \llcorner Z$ for some constant c.

Plan of the chapter. Before going into the details of the various proofs, we briefly outline the plan of this chapter. In the first section we prove Proposition 3.4 and Lemma 3.6. The second section contains Lemma 3.7 and some basic remarks. Lemma 3.8 and Corollary 3.9 are proved, respectively, in the third and fourth section.

1. Tangent measures and Proposition 3.4

Tangent measures can be viewed as a suitable generalization of the concept of tangent planes to a C^1 submanifold of $\mathbb{R}^n$. Indeed, let Γ be a k–dimensional C^1 submanifold of $\mathbb{R}^n$, set $\mu := \mathscr{H}^k \llcorner \Gamma$ and consider $x \in \Gamma$. Then it is not difficult to verify that the measures $r^{-k}\mu_{x,r}$ are given by

$$r^{-k}\mu_{x,r} \;=\; \mathscr{H}^k \llcorner \left(\frac{\Gamma - x}{r}\right) .$$

Here $\Gamma_r := (\Gamma - x)/r$ is the set

$$\left\{y \;:\; ry + x \in \Gamma\right\} .$$

Therefore, since the set Γ is C^1, as $r \downarrow 0$ the sets Γ_r look almost like the tangent plane T_x to Γ at x (see Figure 1). In the next chapter, using the area formula (which relates the abstract definition of Hausdorff measure with the usual differential geometric formula for the volume of a smooth submanifold) we will prove that

$$\mathscr{H}^k \llcorner \Gamma_r \;\stackrel{*}{\rightharpoonup}\; \mathscr{H}^k \llcorner T_x$$

(cp. with Theorem 4.8 and its proof). This implies that $\mathrm{Tan}_k(\mu, x) = \{\mathscr{H}^k \llcorner T_x\}$, as one would naturally expect.

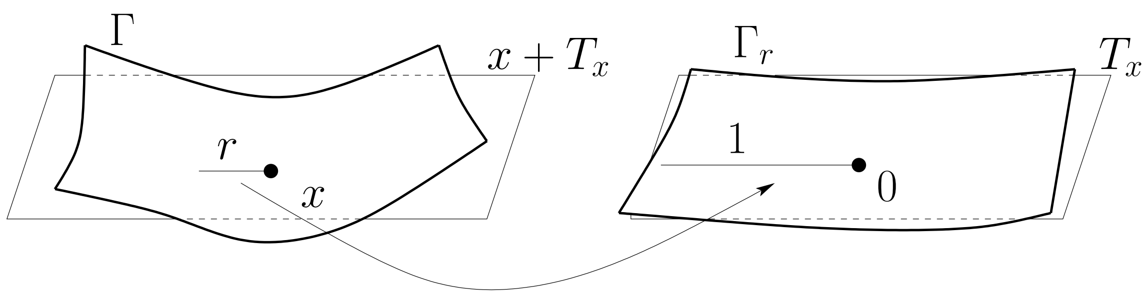

FIGURE 1. From Γ to $\Gamma_r := \{y \;:\; y + rx \in \Gamma\}$

If f is a continuous function and μ a measure, it follows directly from the definition that $\mathrm{Tan}_k(f\mu, x) = f(x)\mathrm{Tan}_k(\mu, x)$. By this we mean that ν belongs to $\mathrm{Tan}_k(f\mu, x)$ if and only if $\nu = f(x)\zeta$ for some $\zeta \in \mathrm{Tan}_k(\mu, x)$. By Proposition 2.2, we can generalize this fact in the following useful proposition:

PROPOSITION 3.12 (Locality of $\mathrm{Tan}_\alpha(\mu, x)$). *Let μ be a measure on $\mathbb{R}^n$ and $f \in L^1(\mu)$ a Borel nonnegative function. Then*

$$\mathrm{Tan}_\alpha(f\mu, x) \;=\; f(x)\mathrm{Tan}_\alpha(\mu, x) \qquad \textit{for } \mu\text{–a.e. } x. \tag{3.8}$$

REMARK 3.13. *As a corollary of Proposition 3.12 we obtain that, for every Borel set B,*

$$\mathrm{Tan}_\alpha(\mu \llcorner B, x) \;=\; \mathrm{Tan}_\alpha(\mu, x) \qquad \textit{for } \mu\text{–a.e. } x \in B. \tag{3.9}$$

Proof of Proposition 3.12. We claim that the equality (3.8) holds for every point x in the set

$$B_1 \ := \ \left\{ x \in \mathbb{R}^n \ : \ \lim_{r \downarrow 0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| \, d\mu(y) \ = \ 0 \right\} , \tag{3.10}$$

and we recall that $\mu(\mathbb{R}^n \setminus B_1) = 0$ (see Proposition 2.2).

To prove the claim, fix $x \in B_1$ and $\nu \in \mathrm{Tan}_\alpha(\mu, x)$. Consider $r_i \downarrow 0$ such that

$$\nu_i \ := \ \frac{\mu_{x,r_i}}{r_i^\alpha} \ \stackrel{*}{\rightharpoonup} \ \nu \, . \tag{3.11}$$

If we define

$$\nu_i' \ := \ \frac{(f\mu)_{x,r_i}}{r_i^\alpha} \, ,$$

then for every ball B_ρ we have

$$\begin{aligned} |f(x)\nu_i - \nu_i'|(B_\rho) \ &\leq \ \frac{1}{r_i^\alpha} \int_{B_{\rho r_i}} |f(y) - f(x)| \, d\mu(x) \\ &= \ \left[\frac{\mu(B_{\rho r_i}(x))}{r_i^\alpha} \right] \frac{1}{\mu(B_{\rho r_i}(x))} \int_{B_{\rho r_i}} |f(y) - f(x)| \, d\mu(x). \end{aligned} \tag{3.12}$$

Note that the quantity

$$\frac{1}{\mu(B_{\rho r_i}(x))} \int_{B_{\rho r_i}} |f(y) - f(x)| \, d\mu(x)$$

vanishes because $x \in B_1$, whereas the ratio

$$\frac{\mu(B_{\rho r_i}(x))}{r_i^\alpha}$$

is bounded because of (3.11). Therefore, we conclude $|f(x)\nu_i - \nu_i'|(B_\rho) \to 0$ for every $\rho > 0$, and hence $\nu_i' \stackrel{*}{\rightharpoonup} f(x)\nu$. This implies $\mathrm{Tan}_\alpha(f\mu, x) \subset f(x)\mathrm{Tan}_\alpha(\mu, x)$. The opposite inclusion follows from a similar argument. $\square$

We are now ready to attack Proposition 3.4, which we prove using a common "countable decomposition" argument.

Proof of Proposition 3.4.

Step 1 For every positive $i, j, k \in \mathbb{N}$, consider the sets

$$E^{i,j,k} \ := \ \left\{ x \ : \ \frac{(j-1)\omega_\alpha}{i} \leq \frac{\mu(B_r(x))}{r^\alpha} \leq \frac{(j+1)\omega_\alpha}{i} \quad \text{for all } r \leq \frac{1}{k} \right\} .$$

Clearly, for every i we have

$$E \ \subset \ \bigcup_{j,k} E^{i,j,k} \, . \tag{3.13}$$

We claim that for μ–a.e. $x \in E^{i,j,k}$ the following holds:

- For every $\nu \in \mathrm{Tan}_\alpha(\mu \llcorner E^{i,j,k}, x)$ we have the estimate

$$\left| \nu(B_r(y)) - \theta^\alpha(\mu, x)\omega_\alpha r^\alpha \right| \ \leq \ \frac{2\omega_\alpha}{i} \qquad \text{for every } y \in \mathrm{supp}\,(\nu) \text{ and } r > 0. \tag{3.14}$$

We will prove the claim in the next step. Note that combining this claim with Remark 3.13 we can conclude that

- If we fix i, then for every j and k and for μ–a.e. $x \in E^{i,j,k}$, the bound of (3.14) holds for every $\nu \in \mathrm{Tan}_\alpha(\mu, x)$.

From (3.13) we conclude that for μ–a.e. $x \in E$, the bound (3.14) holds for every $\nu \in \mathrm{Tan}_\alpha(\mu, x)$. Since i varies in the set of positive integers, which is countable, we conclude that for μ–a.e. $x \in E$, the bound (3.14) holds for every $\nu \in \mathrm{Tan}_\alpha(\mu, x)$ and for every i. Therefore, we conclude that, for any such x and any such ν,

$$\nu(B_r(y)) \;=\; \theta^\alpha(\mu, x)\omega_\alpha r^\alpha \qquad \text{for every } y \in \mathrm{supp}\,(\nu) \text{ and } r > 0.$$

This means that $\nu/\theta^\alpha(\mu, x)$ is an α–uniform measure. To conclude that $\nu/\theta^\alpha(\mu, x) \in \mathcal{U}^\alpha(\mathbb{R}^n)$, it suffices to show that $0 \in \mathrm{supp}\,(\nu)$. This is trivial. Let us argue by contradiction and assume $\nu(\overline{B}_\rho(x)) = 0$. Fix a sequence $r_i \downarrow 0$ such that

$$r_i^{-\alpha}\mu_{x,r_i} \;\stackrel{*}{\rightharpoonup}\; \nu\,.$$

Then we would conclude

$$\limsup_{i\to\infty} \frac{\mu(B_{\rho r_i}(x))}{(\rho r_i)^\alpha} \;=\; \rho^{-\alpha}\lim_{i\to\infty} r_i^{-\alpha}\mu_{x,r_i}(B_\rho) \;=\; \rho^{-\alpha}\nu(B_\rho) \;=\; 0$$

which contradicts $\theta^\alpha_*(\mu, x) > 0$.

Step 2 We are left with the task of proving (3.14) for μ–a.e. $x \in E^{i,j,k}$. To simplify the notation we set

$$F \;:=\; E^{i,j,k} \qquad F_1 \;:=\; \left\{x \in F : \lim_{r\downarrow 0} \frac{\mu(B_r(x) \setminus F)}{r^\alpha} \;=\; 0\right\}.$$

By Proposition 2.2 we have $\mu(F \setminus F_1) = 0$ and therefore it suffices to prove (3.14) when $x \in F_1$. Therefore, we fix $x \in F_1$, $\nu \in \mathrm{Tan}_\alpha(\mu \llcorner F, x)$, and $r_i \downarrow 0$ such that

$$\nu^i \;:=\; \frac{(\mu \llcorner F)_{x,r_i}}{r_i^\alpha} \;\stackrel{*}{\rightharpoonup}\; \nu\,.$$

Note that for every $y \in \mathrm{supp}\,(\nu)$, there exists $\{x_i\} \subset F$ such that

$$y_i \;:=\; \frac{x_i - x}{r_i} \;\to\; y\,.$$

Indeed, if this were not the case, then we would have $\mu_{x,r_{i(k)}}(B_\rho(y)) = 0$ for some $\rho > 0$ and some subsequence $\{r_{i(k)}\}$, which would imply $\nu(B_\rho(y)) = 0$. We claim that there exists $S \subset \mathbb{R}$ at most countable such that

$$\lim_{i\uparrow\infty} \nu^i(B_\rho(y_i)) \;=\; \nu(B_\rho(y)) \qquad \text{for every } \rho \in \mathbb{R}^+ \setminus S. \tag{3.15}$$

Indeed, if we define $\zeta^i \;:= \nu^i_{y_i - y, 1}$, we obtain that $\zeta_i \stackrel{*}{\rightharpoonup} \nu$ and (3.15) translates into

$$\lim_{i\uparrow\infty} \zeta^i(B_\rho(y)) \;=\; \nu(B_\rho(y))\,.$$

Hence, the existence of the countable set S follows from Proposition 2.7.

Let us compute

$$\lim_{i\to\infty} \nu^i(B_\rho(y_i)) \;=\; \lim_{i\to\infty} \frac{\mu(B_{\rho r_i}(x_i)\cap F)}{r_i^\alpha} \;=\; \lim_{i\to\infty} \frac{\mu(B_{\rho r_i}(x_i))}{r_i^\alpha}\,.$$

From this and from the definition of F, we conclude that (3.14) holds for every $\rho \in \mathbb{R}^+ \setminus S$. Since S is countable, for every $\rho \in S$ there exists $\{\rho_j\} \subset \mathbb{R}^+ \setminus S$ with $\rho_j \uparrow \rho$. Hence, $\nu(B_\rho(y)) = \lim_j \nu(B_{\rho_j}(y))$ and from this we conclude that (3.14) is valid for every $r \in \mathbb{R}^+$.

Step 3 So far we have proved that

$$\mathrm{Tan}_\alpha(\mu,x) \;\subset\; \theta^\alpha(\mu,x)\,\mathcal{U}^\alpha(\mathbb{R}^n) \qquad \text{for } \mu\text{–a.e. } x\in\mathbb{R}^n.$$

It remains to show that for μ–a.e. $x \in \mathbb{R}^n$ the set $\mathrm{Tan}_\alpha(\mu,x)$ is not empty. Let us fix any x such that $\theta^{\alpha *}(\mu,x) < \infty$. Then, for every $\rho > 0$, the set of numbers

$$r^{-\alpha}\mu(B_{\rho r}(x)) \;=\; r^{-\alpha}\mu_{x,r}(B_\rho) \qquad r\le 1$$

is uniformly bounded. Therefore, the family of measures $\{r^{-\alpha}\mu_{x,r}\}_{r\le 1}$ is locally uniformly bounded. From the compactness of the weak* topology of measures, it follows that there exists a sequence $r_j \downarrow 0$ and a measure μ_∞ such that $\mu_{x,r_j} \overset{*}{\rightharpoonup} \mu_\infty$. Hence, $\mu_\infty \in \mathrm{Tan}_\alpha(\mu,x)$. $\square$

Proof of Lemma 3.6. In this case, the argument given in Step 3 of the proof of Proposition 3.4 shows that $\mathrm{Tan}_\alpha(\mu,x) \neq \emptyset$ for every $x \in \mathrm{supp}\,(\mu)$.

Now fix any $x \in \mathrm{supp}\,(\mu)$ and any $\nu \in \mathrm{Tan}_\alpha(\mu,x)$, and let $r_i \downarrow 0$ be such that

$$r_i^{-\alpha}\mu_{x,r_i} \;\overset{*}{\rightharpoonup}\; \nu\,.$$

Given any $y \in \mathrm{supp}\,(\nu)$, we argue as in Step 2 of the proof of Proposition 3.4 in order to conclude that

- There exists a sequence $\{x_i\} \subset \mathrm{supp}\,(\mu)$ such that

$$y_i \;:=\; \frac{x_i - x}{r_i} \;\to\; y;$$

- There exists a countable set $S \subset \mathbb{R}^+$ such that

$$\lim_{i\uparrow\infty} r_i^{-\alpha}\mu_{x,r_i}(B_\rho(y_i)) \;=\; \nu(B_\rho(y)) \qquad \text{for every } \rho\in\mathbb{R}^+\setminus S.$$

Thus, for every $\rho \in \mathbb{R}^+ \setminus S$ we have

$$\nu(B_\rho(y)) \;=\; \lim_{r_i\downarrow 0} \frac{\mu(B_{\rho r_i}(x_i))}{r_i^\alpha} \;=\; \omega_\alpha \rho^\alpha\,.$$

For every $\rho \in \mathbb{R}^+$ there exists a sequence $\{\rho_j\} \subset \mathbb{R}^+ \setminus S$ such that $\rho_j \uparrow \rho$. Therefore, we conclude that $\nu(B_\rho(y)) = \omega_\alpha\rho^\alpha$ for every $\rho > 0$. The arbitrariness of $y \in \mathrm{supp}\,(\nu)$ implies $\nu \in \mathcal{U}^\alpha(\mathbb{R}^n)$. $\square$

2. Lemma 3.7 and some easy remarks

REMARK 3.14. *Assume that $\mu \in \mathcal{U}^\alpha(\mathbb{R}^n)$. If $\alpha \geq n$, from the Besicovitch Differentiation Theorem we conclude that $\mu = f\mathscr{L}^n$, where*

$$f(x) \;=\; \lim_{r\downarrow 0} \frac{\mu(B_r(x))}{\omega_n r^n} \qquad \text{for } \mathscr{L}^n\text{–a.e. } x.$$

If $\alpha > n$, we conclude that $f = 0$. If $\alpha = n$ we obtain that $f = \mathbf{1}_E$, where $E = \mathrm{supp}\,(\mu)$. Since $\mu \in \mathcal{U}^n(\mathbb{R}^n)$, we conclude that $\mathscr{L}^n(B_r(0)\cap E) = \omega_n r^n = \mathscr{L}^n(B_r(0))$. Since E is closed, we obtain $B_r(0) \subset E$ and the arbitrariness of r implies $E = \mathbb{R}^n$.

Combining this argument with Proposition 2.13, we conclude that: If $\mu \in \mathcal{U}^m(\mathbb{R}^n)$ and $\mathrm{supp}\,(\mu)$ is contained in an m–dimensional linear plane V, then $\mu = \mathscr{H}^m \llcorner V$.

PROOF OF LEMMA 3.7. Set $E := \mathrm{supp}\,(\mu)$ and note that, since $\alpha < n$, $B_1(0) \not\subset E$. Indeed, we can use the Besicovitch–Vitali Covering Theorem to cover $\mathscr{L}^n$–almost all $B_1(0)$ with a collection of pairwise disjoint balls $\{B_{r_j}(x_j)\}$ contained in $B_1(0)$ and with radii strictly less than 1. If we had $B_1(0) \subset E$ then we could estimate

$$\begin{aligned}\mu(B_1(0)) \;&\geq\; \sum_j \mu(B_{r_j}(x_j)) \;=\; \omega_\alpha \sum_j r_j^\alpha \;>\; \omega_\alpha \sum_j r_j^n \;=\; \frac{\omega_\alpha}{\omega_n} \sum_j \mathscr{L}^n(B_{r_j}(x_j))\\ &=\; \frac{\omega_\alpha}{\omega_n} \mathscr{L}^n(B_1(0)) \;=\; \omega_\alpha\,,\end{aligned}$$

which contradicts $\mu(B_1(0)) = \omega_\alpha$

Fix $y \notin E$. Since E is a nonempty closed set, there exists $z \in E$ such that $\mathrm{dist}\,(y, E) = |y - z| =: a$. Without loss of generality, we take z to be the origin and we fix a system of coordinates $x_1, \ldots, x_n$ such that $y = (-a, 0, \ldots, 0)$. Clearly, E is contained in the closed set

$$\tilde{E} \;:=\; \mathbb{R}^n \setminus B_a(y) \;=\; \left\{x : (a + x_1)^2 + x_2^2 + \ldots + x_n^2 \geq a^2\right\}.$$

Fix $\nu \in \mathrm{Tan}_\alpha(\mu, 0)$ and a sequence $r_i \downarrow 0$ such that

$$\nu_i \;:=\; \frac{\mu_{0,r_i}}{r_i^\alpha} \;\stackrel{*}{\rightharpoonup}\; \nu\,.$$

The support of ν_i is given by

$$E_i \;:=\; E/r_i \;\subset\; \tilde{E}_i \;:=\; \left\{(a + r_i x_1)^2 + r_i^2(x_2^2 + \ldots + x_n^2) \;\geq\; a^2\right\}.$$

Note that, for any $x \in \{x_1 < 0\}$, there exists $N > 0$ such that $x \notin \tilde{E}_i$ for $i \geq N$; cf. Figure 2. This implies that $\mathrm{supp}\,(\nu) \subset \{x_1 \geq 0\}$ and concludes the proof. $\Box$

3. Proof of Lemma 3.8

REMARK 3.15. *Let $\mu \in \mathcal{U}^\alpha(\mathbb{R}^n)$ and $f : \mathbb{R}^+ \to \mathbb{R}$ be a simple function, that is $f(t) = \sum_{i=1}^N a_i \mathbf{1}_{[0,r_i[}$ for some choice of $N \in \mathbb{N}$, $r_i > 0$ and $a_i \in \mathbb{R}$. Then, for any $y \in \mathrm{supp}\,(\mu)$ we have*

$$\int f(|z|)\, d\mu(z) \;=\; \sum_{i=1}^N a_i \mu(B_{r_i}(0)) \;=\; \sum_{i=1}^N a_i \mu(B_{r_i}(y)) \;=\; \int f(|z - y|)\, d\mu(z)\,.$$

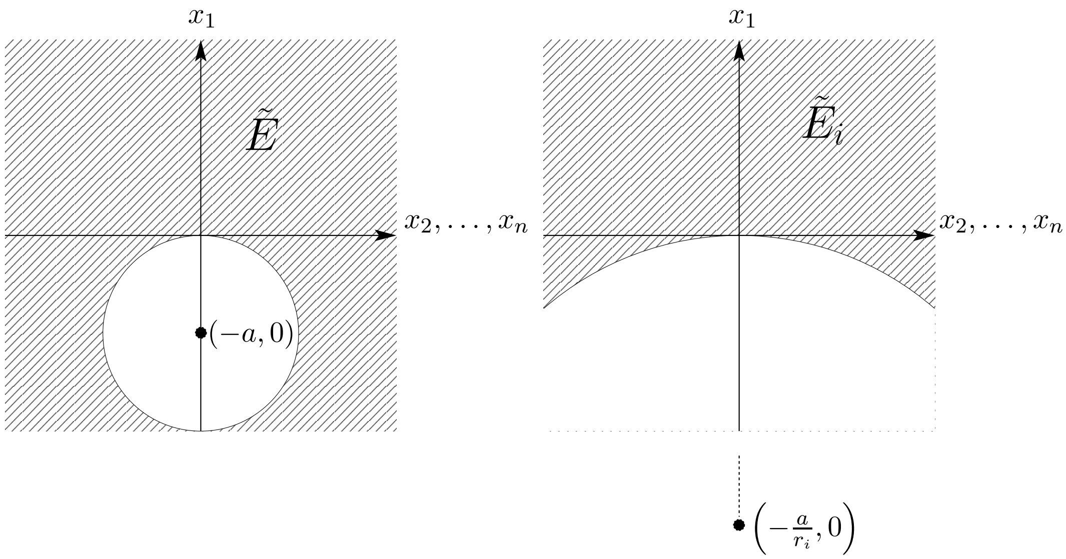

FIGURE 2. The sets $\tilde{E}_i$ converge to the closed upper half–space.

By a simple approximation argument we conclude that

$$\int \varphi(z)\, d\mu(z) \;=\; \int \varphi(z-y)\, d\mu(z) \qquad \textit{for any radial } \varphi \in L^1(\mu) \textit{ and } \forall y \in \operatorname{supp}(\mu). \quad (3.16)$$

PROOF. Let us define the quantity

$$b(r) \;:=\; \frac{\omega_\alpha}{\nu(B_r(0))} \int_{B_r(0)} z\, d\nu(z) \;=\; r^{-\alpha} \int_{B_r(0)} z\, d\nu(z)\,, \tag{3.17}$$

(in other words, $b(r)$ is given by ω_α times the barycenter of the measure $\nu \llcorner B_r(0)$). We let $(b_1(r), \ldots, b_n(r))$ be the components of the vector $b(r)$.

Since $\operatorname{supp}(\nu) \subset \{x_1 \geq 0\}$, we have $b_1(r) \geq 0$. Moreover, $b_1(r) = 0$ would imply that $\operatorname{supp}(\nu) \subset \{x_1 = 0\}$ and the claim of the lemma would follow trivially. The idea is to study the limiting behavior of $b(r)$ as $r \downarrow 0$. More precisely, given $\tilde{\nu} \in \operatorname{Tan}_\alpha(\nu, 0)$, we define

$$c(r) \;:=\; r^{-\alpha} \int_{B_r(0)} z\, d\tilde{\nu}(z)\,. \tag{3.18}$$

Our goal is to show that $c(r) = 0$ for every r. Since $\operatorname{supp}(\tilde{\nu}) \subset \{x_1 \geq 0\}$, this would imply $\operatorname{supp}(\tilde{\nu}) \subset \{x_1 = 0\}$ and conclude the proof of the lemma.

Step 1 In this step we prove the following claim:

$$|\langle b(r), y\rangle| \;\leq\; C(\alpha)|y|^2 \qquad \text{for every } y \in \operatorname{supp}(\nu) \cap B_{2r}(y). \tag{3.19}$$

Using the identity

$$2\langle x, y\rangle \;=\; |y|^2 + \left(r^2 - |x-y|^2\right) - \left(r^2 - |x|^2\right),$$

we can compute

$$\begin{aligned} 2\big|\langle b(r), y\rangle\big| &= r^{-\alpha}\left|\int_{B_r(0)} 2\langle x, y\rangle\, d\nu(x)\right| \\ &= r^{-\alpha}\left||y|^2\nu(B_r(0)) + \int_{B_r(0)}\big(r^2-|x-y|^2\big)\,d\nu(x) - \int_{B_r(0)}\big(r^2-|x|^2\big)\,d\nu(x)\right|. \end{aligned} \tag{3.20}$$

For $y \in \operatorname{supp}(\nu)$, Remark 3.15 gives

$$\begin{aligned} &\int_{B_r(0)}\big(r^2-|x-y|^2\big)\,d\nu(x) - \int_{B_r(0)}\big(r^2-|x|^2\big)\,d\nu(x) \\ &= \int_{B_r(0)}\big(r^2-|x-y|^2\big)\,d\nu(x) - \int_{B_r(y)}\big(r^2-|x-y|^2\big)\,d\nu(x) \\ &= \int_{B_r(0)\setminus B_r(y)}\big(r^2-|x-y|^2\big)\,d\nu(x) - \int_{B_r(y)\setminus B_r(0)}\big(r^2-|x-y|^2\big)\,d\nu(x)\,. \end{aligned} \tag{3.21}$$

Combining (3.20) and (3.21) we obtain

$$\begin{aligned} 2|\langle b(r), y\rangle| &\le \omega_\alpha|y|^2 + r^{-\alpha}\int_{B_r(0)\setminus B_r(y)}\big|r^2-|x-y|^2\big|\,d\nu(x) \\ &\quad + r^{-\alpha}\int_{B_r(y)\setminus B_r(0)}\big|r^2-|x-y|^2\big|\,d\nu(x)\,. \end{aligned} \tag{3.22}$$

For $x \in B_r(0)\setminus B_r(y)$, we have

$$0 \le |x-y|^2 - r^2 \le |x-y|^2 - |x|^2 = \big(|x-y|+|x|\big)\big(|x-y|-|x|\big) \le 3r|y|\,,$$

whereas for $x \in B_r(y)\setminus B_r(0)$ we have

$$0 \le r^2 - |x-y|^2 \le |x|^2 - |x-y|^2 = \big(|x-y|+|x|\big)\big(|x|-|x-y|\big) \le 3r|y|\,.$$

Hence, (3.22) gives

$$\begin{aligned} 2|\langle b(r), y\rangle| &\le \omega_\alpha|y|^2 + \frac{3r|y|}{r^\alpha}\Big[\nu\big(B_r(y)\setminus B_r(0)\big) + \nu\big(B_r(0)\setminus B_r(y)\big)\Big] \\ &= \omega_\alpha|y|^2 + \frac{3r|y|}{r^\alpha}\nu\Big[\big(B_r(y)\setminus B_r(0)\big)\cup\big(B_r(0)\setminus B_r(y)\big)\Big]\,. \end{aligned} \tag{3.23}$$

Clearly, if $|y| \le r$, then

$$\big(B_r(y)\setminus B_r(0)\big)\cup\big(B_r(0)\setminus B_r(y)\big) \subset B_{r+|y|}(0)\setminus B_{r-|y|}(y)\,.$$

Hence

$$2|\langle b(r), y\rangle| \le \omega_\alpha|y|^2 + \frac{3|y|}{r^{\alpha-1}}\Big[\nu\big(B_{r+|y|}(0)\big) - \nu\big(B_{r-|y|}(y)\big)\Big] \tag{3.24}$$

$$= \omega_\alpha|y|^2 + \frac{3|y|\omega_\alpha}{r^{\alpha-1}}\Big[(r+|y|)^\alpha - (r-|y|)^\alpha\Big] \le C(\alpha)|y|^2\,. \tag{3.25}$$

This gives (3.19) for $|y| \le r$. For $r \le |y| \le 2r$ we use

$$B_r(y)\setminus B_r(0)\cup B_r(0)\setminus B_r(y) \subset B_{r+|y|}(0)$$

and a similar computation.

Step 2 To reach the desired conclusion, fix $\tilde{\nu} \in \mathrm{Tan}_\alpha(\nu, 0)$ and a sequence $r_i \downarrow 0$ such that

$$\nu^i := \frac{\nu_{0,r_i}}{r_i^\alpha} \overset{*}{\rightharpoonup} \tilde{\nu}\,.$$

Moreover, let $b(r)$ and $c(r)$ be the quantities defined in (3.17) and (3.18). By Proposition 2.7, there is a set $S \subset \mathbb{R}$ which is at most countable and such that

$$c(\rho) = \lim_{r_i \downarrow 0} b(\rho r_i) \qquad \text{for } \rho \in \mathbb{R}^+ \setminus S.$$

Let $\rho \in \mathbb{R}^+ \setminus S$ and $z \in \operatorname{supp}(\tilde{\nu}) \cap B_\rho(0)$. Then there exists a sequence $\{z_i\}$ converging to z such that $y_i := r_i z_i \in \operatorname{supp}(\nu)$. Using (3.19), we obtain

$$|\langle c(\rho), z\rangle| = \lim_{r_i \downarrow 0} \frac{|\langle b(\rho r_i), y_i\rangle|}{r_i} \le C(\alpha) \lim_{r_i \downarrow 0} \frac{|y_i|^2}{r_i} = 0\,.$$

This means that $\langle c(\rho), z\rangle = 0$ for every $z \in \operatorname{supp}(\tilde{\nu}) \cap B_\rho(0)$. Therefore

$$0 = \rho^{-\alpha} \int_{B_\rho(0)} \langle c(\rho), z\rangle \, d\tilde{\nu}(z) = |c(\rho)|^2\,.$$

This holds for every $\rho \in \mathbb{R}^+ \setminus S$. Since for $\rho \in S$ there exists $\{\rho_i\} \subset \mathbb{R}^+ \setminus S$ with $\rho_i \uparrow \rho$, we conclude

$$c(\rho) = \lim_{i \to \infty} c(\rho_i) = 0\,,$$

and hence $c(\rho) = 0$ for every $\rho > 0$, which completes the proof. □

4. Proof of Corollary 3.9

Proof. For every measure μ, we denote by $\mathcal{T}_\alpha(\mu)$ the weak* closure of the set

$$\left\{\frac{\mu_{x,r}}{r^\alpha} \,:\, x \in \mathbb{R}^n, r \in \mathbb{R}^+\right\}\,.$$

Note that for every x we have $\mathrm{Tan}_\alpha(\mu, x) \subset \mathcal{T}_\alpha(\mu)$. Let m and μ be as in the statement of the lemma and set $k := n - m$. We apply Lemmas 3.6, 3.7, and 3.8 to find a family of measures $\{\mu_i\}_{i \in \{0,1,\dots,2k\}} \subset \mathcal{U}^m(\mathbb{R}^n)$ such that

- $\mu_0 = \mu$ and $\mu_{i+1} \in \mathcal{T}_\alpha(\mu_i)$;
- $\operatorname{supp}(\mu_{2k}) \subset V$ for some m–dimensional linear plane $V \subset \mathbb{R}^n$.

From Remark 3.14 it follows that $\mu_{2k} = \mathscr{H}^m \llcorner V$, and hence the corollary follows if we can prove $\mu_{2k} \in \mathcal{T}_\alpha(\mu_0)$. To show this, it suffices to apply the following claim $2k$ times:

$$\xi \in \mathcal{T}_\alpha(\nu) \qquad \Longrightarrow \qquad \mathcal{T}_\alpha(\xi) \subset \mathcal{T}_\alpha(\nu)\,. \tag{3.26}$$

By the weak* closure of $\mathcal{T}_\alpha(\nu)$, this claim is equivalent to

$$\xi \in \mathcal{T}_\alpha(\nu) \qquad \Longrightarrow \qquad \frac{\xi_{x,\rho}}{\rho^\alpha} \in \mathcal{T}_\alpha(\nu)\,. \tag{3.27}$$

Fix ξ as in (3.27). Then there are sequences $\{x_i\}$ and $\{r_i\}$ such that

$$\nu^i := \frac{\nu_{x_i,r_i}}{r_i^\alpha} \overset{*}{\rightharpoonup} \xi\,.$$

Clearly $\nu^i_{x,\rho} \overset{*}{\rightharpoonup} \xi_{x,\rho}$. Hence, if we define $z_i := x_i + r_i x$ and $\rho_i := \rho r_i$, we conclude that

$$\frac{\nu_{z_i,\rho_i}}{\rho_i^\alpha} = \frac{\nu^i_{x,\rho}}{\rho^\alpha} \overset{*}{\rightharpoonup} \frac{\xi_{x,\rho}}{\rho^\alpha}\,.$$

This proves the desired claim (3.27) and completes the proof. □

CHAPTER 4

Rectifiability

This chapter deals with rectifiable sets and rectifiable measures.

DEFINITION 4.1 (Rectifiability). *A k–dimensional Borel set $E \subset \mathbb{R}^n$ is called* rectifiable *if there exists a countable family $\{\Gamma_i\}_i$ of k–dimensional Lipschitz graphs such that $\mathscr{H}^k\big(E \setminus \bigcup \Gamma_i\big) = 0$.*

A measure μ is called a k–dimensional rectifiable measure *if there exist a k–dimensional rectifiable set E and a Borel function f such that $\mu = f\mathscr{H}^k \llcorner E$.*

By the Whitney Extension Theorem, we could replace Lipschitz with C^1 in the previous definition. However we will never use this fact in these notes. The final goal of this chapter is to give a first characterization of rectifiable measures in terms of their tangent measures; see Theorem 4.8.

The Area Formula. When E is rectifiable we have an important tool which relates the abstract definition of $\mathscr{H}^k(E)$ to the differential geometric formula commonly used to compute the k–volume of a C^1 manifold. This tool is the area formula.

DEFINITION 4.2 (Jacobian determinant). *Let $A \in \mathbb{R}^{m\times k}$ be a matrix and $L(x) = A \cdot x$ the linear map $L : \mathbb{R}^k \to \mathbb{R}^m$ naturally associated to it. We define $JL := (\det{(A^t \cdot A)})^{1/2}$.*

Let G be a Borel set and $f : G \to \mathbb{R}^m$ a Lipschitz map. We denote by df_x the differential of f at the point x which, thanks to Rademacher's Theorem, exists at $\mathscr{L}^k$–a.e. $x \in G$. We denote by Jf the Borel function $Jf(x) := Jdf_x$.

PROPOSITION 4.3 (Area Formula). *Let $E \subset \mathbb{R}^k$ be a Borel set and $f : E \to \mathbb{R}^n$ a Lipschitz map. Then*

$$\int_{f(E)} \mathscr{H}^0\big(f^{-1}(\{z\})\big)\, d\mathscr{H}^k(z) \;=\; \int_E Jf(x)\, d\mathscr{L}^k(x)\,. \tag{4.1}$$

Recall that $\mathscr{H}^0(F)$ gives the number of elements of F. When Γ is a Borel subset of a k–dimensional Lipschitz graph, there exists a Lipschitz function $f : \mathbb{R}^k \to \mathbb{R}^{n-k}$ and a Borel set E such that $\Gamma = \{(x, f(x)) : x \in E\}$. Therefore, we can apply the previous proposition to the Lipschitz map

$$F \;:\; \mathbb{R}^k \ni x \;\to\; (x, f(x)) \in \mathbb{R}^n\,,$$

in order to obtain

$$\mathscr{H}^k(\Gamma) \;=\; \int_E JF(x)\, d\mathscr{L}^k(x)\,.$$

If we fix a point x where f is differentiable and df is Lebesgue continuous, then:

- $JF(y)$ will be close to Jdf_x for most points y close to x;
- Close to x, Γ will look very much like the plane tangent to Γ at x.

Therefore, it is not surprising that the following corollary holds.

Corollary 4.4. *Let μ be a k–dimensional rectifiable measure. Then for μ–a.e. y there exist a positive constant c_y and a k–dimensional linear plane V_y such that*

$$\mathrm{Tan}_k(\mu, y) \;=\; \left\{c_y \mathscr{H}^k \llcorner V_y\right\}.$$

The Rectifiability Criterion. We now come an important question: How does one prove that a set is rectifiable? The most common tool used for this purpose is the criterion given by Proposition 4.6. Before stating it, we introduce some notation.

Definition 4.5 (k–cones). *Let V be a k–dimensional linear plane of $\mathbb{R}^n$. Then we denote by $V^\perp$ the orthogonal complement of V. Moreover, we denote by P_V and Q_V respectively the orthogonal projection on V and $V^\perp$. For $\alpha \in \mathbb{R}^+$, we denote by $C(V,\alpha)$ the set*

$$\left\{x \in \mathbb{R}^n \;:\; |Q_V(x)| \leq \alpha |P_V(x)|\right\}.$$

For every $x \in \mathbb{R}^n$, we denote by $C(x,V,\alpha)$ the set $x + C(V,\alpha)$. Any such $C(x,V,\alpha)$ will be called a k–cone centered at x.

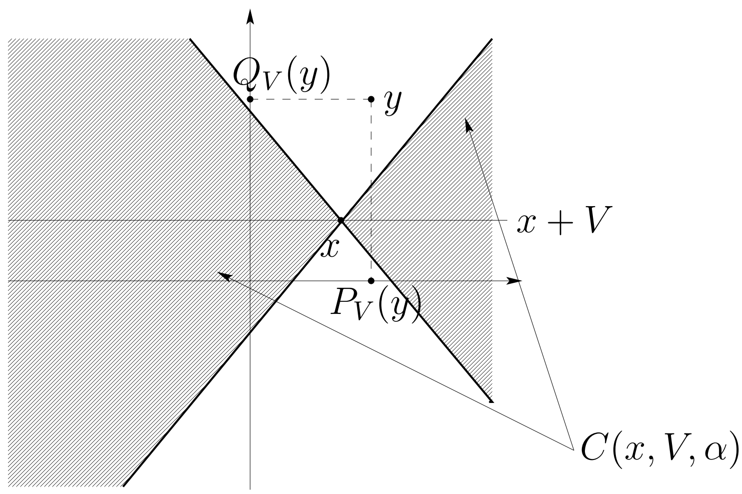

Figure 1. The cone $C(x,V,\alpha)$.

Proposition 4.6 (Rectifiability Criterion). *Let $E \subset \mathbb{R}^n$ be a Borel set such that $0 < \mathscr{H}^k(E) < \infty$. Assume the following two conditions hold for $\mathscr{H}^k$–a.e. $x \in E$:*

- $\theta^k_*(E,x) > 0$*;*
- *There exists a k–cone $C(x,V,\alpha)$ such that*

$$\lim_{r\downarrow 0} \frac{\mathscr{H}^k\big(E \cap B_r(x) \setminus C(x,V,\alpha)\big)}{r^k} \;=\; 0\,. \tag{4.2}$$

Then E is rectifiable.

The main idea behind Proposition 4.6 is that the conditions above are some sort of approximate version of (4.3) below. Indeed, the proof of Proposition 4.6 uses the following elementary geometric observation, which will also be useful later:

LEMMA 4.7 (Geometric Lemma). *Let $F \subset \mathbb{R}^n$. Assume that there exists a k–dimensional linear plane V and a real number α such that*

$$F \subset C(x, V, \alpha) \qquad \text{for every } x \in F. \tag{4.3}$$

Then there exists a Lipschitz map $f : V \to V^\perp$ such that $F \subset \{(x, f(x)) : x \in V\}$.

The proof of Proposition 4.6 shows that one can decompose $\mathscr{H}^k$–almost all E into a countable union of sets F_i satisfying the assumption of the lemma. In this decomposition the condition $\theta^k_*(x, E) > 0$ will play a crucial role.

First characterization of rectifiable measures. A corollary of Proposition 4.6 is the converse of Corollary 4.4. Therefore, rectifiable measures can be characterized in terms of their tangent measures in the following way:

THEOREM 4.8. *A measure μ is a k–dimensional rectifiable measure if and only if for μ–a.e. x there exists a positive constant c_x and a k–dimensional linear plane V_x such that*

$$\mathrm{Tan}_k(\mu, x) = \left\{c_x \mathscr{H}^k \llcorner V_x\right\}. \tag{4.4}$$

Plan of the chapter. The plan of the chapter is the following: In the first two sections we prove the Area Formula and Corollary 4.4; in the third section we prove the Geometric Lemma and the Rectifiability Criterion; in the fourth section we prove Theorem 4.8.

1. The Area Formula I: Preliminary lemmas

First of all, we check that the Area Formula holds when the map f is affine. Indeed, in this case, $f(E)$ is contained in a k–dimensional affine plane. Thus, after a suitable change of coordinates, the Area Formula becomes the usual formula for changing variables in the Lebesgue integral.

LEMMA 4.9. *Let f in Proposition 4.3 be affine. Then* (4.1) *holds.*

PROOF. Since f is affine, there exists a matrix $A \in \mathbb{R}^{n\times k}$ and a constant $c \in \mathbb{R}^n$ such that $f(x) = c + A \cdot x$. Without loss of generality we assume that $c = 0$. Moreover, note that $Jf = (\det(A^t \cdot A))^{1/2}$.

Clearly $f(E)$ is a subset of some k–dimensional linear plane V and we can find an orthonormal system of coordinates $y_1, \ldots, y_k, y_{k+1}, \ldots, y_n$ such that

$$V = \left\{y_{k+1} = y_{k+2} = \ldots = y_n = 0\right\}.$$

Writing f in this new system of coordinates is equivalent to finding an orthogonal matrix $O \in \mathbb{R}^{n\times k}$ such that

$$f(x) = B \cdot x = O \cdot A \cdot x\,.$$

Clearly $(\det(B^t \cdot B))^{1/2} = (\det(A^t \cdot A))^{1/2}$. We denote by $f_j(x)$ the j–th component of the vector $f(x)$ in the system of coordinates $y_1, \ldots, y_n$ and we define $\tilde{f} : \mathbb{R}^k \to \mathbb{R}^k$ as

$$\tilde{f}(x) = \big(f_1(x), \ldots, f_k(x)\big)\,.$$

Moreover, we define $\iota : \mathbb{R}^k \to \mathbb{R}^n$ by $\iota(z) = (z, 0, \ldots 0)$. Then, according to Proposition 2.13 we have

$$\mathscr{H}^k(F) = \mathscr{L}^k(\iota^{-1}(F)) \qquad \text{for every Borel } F \subset f(E).$$

This implies

$$\int_{f(E)} \mathscr{H}^0\big(f^{-1}(y)\big)\, d\mathscr{H}^k(y) \;=\; \int_{\tilde f(E)} \mathscr{H}^0\big(\tilde f^{-1}(y)\big)\, d\mathscr{L}^k(y)\,. \tag{4.5}$$

Moreover, $(\det(A^t \cdot A))^{1/2} = J\tilde f(x)$, and since $\tilde f$ is a map between spaces of the same dimension, it is easy to check that $J\tilde f = |\det d\tilde f|$. Therefore, the usual formula for the change of variables in the Lebesgue integral yields:

$$\int_{\tilde f(E)} \mathscr{H}^0\big(\tilde f^{-1}(y)\big)\, d\mathscr{L}^k(y) \;=\; \int_E |\det d\tilde f_x|\, d\mathscr{L}^k(x) \;=\; \int_E Jf(x)\, d\mathscr{L}^k(x)\,. \tag{4.6}$$

Combining (4.5) and (4.6) we obtain (4.1). □

The next two lemmas deal with two other relevant cases of the Area Formula: The case where $\mathscr{L}^k(E) = 0$ and the case where $Jf(x) = 0$ for $\mathscr{L}^k$–a.e. x.

Lemma 4.10. *Let f be as in Proposition* (4.3) *and assume $\mathscr{L}^k(E) = 0$. Then* (4.1) *holds.*

Proof. The proof follows trivially from Proposition 2.12(iv). □

Lemma 4.11. *Let f be as in Proposition* (4.3)*. If $Jf(x) = 0$ for $\mathscr{L}^k$–a.e. $x \in E$, then* (4.1) *holds.*

Proof. Clearly we have to show $\mathscr{H}^k(f(E)) = 0$. Let

$$F \;:=\; \Big\{x \in E \;:\; f \text{ is differentiable at } x \text{ and } J(df_x) = 0\Big\}\,.$$

Since $\mathscr{L}^k(E \setminus F) = 0$, we conclude that $\mathscr{H}^k(f(E \setminus F)) = 0$. Therefore, it suffices to prove $\mathscr{H}^k(f(F)) = 0$.

Without loss of generality we may assume that F is contained in the ball $B_R = B_R(0) \subset \mathbb{R}^m$. Moreover, recall that, since f is a Lipschitz map, there exists a constant M such that $|df_x| \le M$ for every $x \in F$. We will prove that

$$\mathscr{H}^k_{2\varepsilon}(f(F)) \;\le\; c\varepsilon \qquad \text{for every } \varepsilon > 0, \tag{4.7}$$

where c is a constant which depends only on M, k, n, and R. Letting $\varepsilon \downarrow 0$, we conclude $\mathscr{H}^k(f(F)) = 0$.

First covering For every $x \in F$, denote by $A_x : \mathbb{R}^k \to \mathbb{R}^n$ the affine map given by $A_x(y) = f(x) + df_x(y - x)$. Since every $x \in F$ is a point of differentiability, there exists a positive $r_x \le 1$ such that:

$$\big|f(y) - A_x(y)\big| \;\le\; \varepsilon|x - y| \qquad \text{for all } y \in B_{r_x}(x)\,. \tag{4.8}$$

Therefore, for every $\rho \le r_x$ we have

$$f(B_\rho(x)) \;\subset\; I^\varepsilon(x,\rho) \;:=\; \Big\{z \in \mathbb{R}^n \;:\; \operatorname{dist}\big(z, A_x(B_\rho(x))\big) \;\le\; \varepsilon\rho\Big\}\,. \tag{4.9}$$

From the $5r$–covering Lemma, we can cover F with a countable family of balls $\{B_{r_i}(x_i)\}$ such that

- $x_i \in F$ and $5r_i \le r_{x_i}$;
- The balls $B_{r_i/5}(x_i)$ are pairwise disjoint and contained in B_R.

Therefore, we conclude that

$$r_i \;\leq\; 1 \qquad \text{and} \qquad \sum_i r_i^k \;\leq\; 5^k R^k\,, \tag{4.10}$$

$$f(F) \;\subset\; \bigcup_i I^\varepsilon(x_i, r_i)\,. \tag{4.11}$$

Second covering Recall that $J(df_{x_i}) = 0$. Therefore, the rank of the linear map df_{x_i} is at most $k-1$ and hence $A_{x_i}(B_{r_i}(x_i))$ is contained in a $(k-1)$–dimensional affine plane V_i. Moreover, $|df_{x_i}| \leq M$. Therefore, $A_{x_i}(B_{r_i}(x_i))$ is contained in a $(k-1)$–dimensional disk $D_i \subset V_i$ of radius Mr_i. Hence

$$I^\varepsilon(x_i, r_i) \;\subset\; \left\{z \in \mathbb{R}^n \;:\; \operatorname{dist}(z, D_i) \;\leq\; \varepsilon r_i\right\}. \tag{4.12}$$

Then, it is elementary to check that each $I^\varepsilon(x_i, r_i)$ can be covered by $C\varepsilon^{-(k-1)}$ n–dimensional balls $B_{i,j}$ of radius εr_i (where the constant C depends only on k, m, and M).

From (4.11) we obtain that $\{B_{i,j}\}$ is a countable covering of F. Moreover, the diameter of each $B_{i,j}$ is precisely $2\varepsilon r_i \leq 2\varepsilon$. Therefore,

$$\mathscr{H}^k_{2\varepsilon}(f(F)) \;\leq\; \omega_k \sum_{i,j} (\varepsilon r_i)^k \;\leq\; \omega_k \sum_i C\varepsilon r_i^k \;\stackrel{(4.10)}{\leq}\; \omega_k C 5^k R^k \varepsilon\,.$$

Since C depends only on k, n, and M, this is the desired inequality (3.26). □

2. The Area Formula II

The intuitive idea behind the proof of the Area Formula is that, after discarding the set $E_0 \subset E$ where f is not differentiable or where the Jacobian determinant is 0, we can cover $E \setminus E_0$ with a countable number of Borel sets $E_i \subset E$ such that on each E_i the map f is very close to an injective affine map. We make this idea more precise in Lemma 4.12 below, where we will use the following notation:

- If the map $f : G \to H$ is injective, then f^{-1} denotes the inverse of $f : G \to f(G)$.

LEMMA 4.12. *Let $E \subset \mathbb{R}^k$ be a Borel set and $f : E \to \mathbb{R}^n$ a Lipschitz map. Fix any $t > 1$. Then there exists a countable covering of E with Borel subsets $\{E_i\}_{i\geq 0}$ such that:*

(i) *If $x \in E_0$, then either f is not differentiable at x, or $Jf(x) = 0$.*
(ii) *For every $i \geq 1$, the map f is injective on E_i.*
(iii) *For every $i \geq 1$ there exists an injective linear map $L_i : \mathbb{R}^k \supset E_i \to \mathbb{R}^n$ such that the following estimates hold:*

$$\operatorname{Lip}\left(f|_{E_i} \circ L_i^{-1}\right) \;\leq\; t \qquad \operatorname{Lip}\left(L_i \circ (f|_{E_i})^{-1}\right) \;\leq\; t\,, \tag{4.13}$$

$$t^{-n} JL_i \;\leq\; Jf(x) \;\leq\; t^n JL_i \qquad \forall x \in E_i\,. \tag{4.14}$$

PROOF. We define E_0 as the set of points $x \in E$ where f is not differentiable or $Jf(x) = 0$ (i.e. df_x is not injective).

Next we fix:

- $\varepsilon > 0$ such that $t^{-1} + \varepsilon < 1 < t - \varepsilon$;
- $C \subset E$ dense and countable;
- S dense and countable in the set of injective linear maps $L : \mathbb{R}^k \to \mathbb{R}^n$.

For every $x \in C$, $L \in S$, and $i \geq 1$ we define:

$$E(x,L,i) = \Big\{ y \in B_{1/i}(x) : f \text{ is differentiable at } y,\ df_y \text{ is injective, (4.15) and (4.16) hold} \Big\} ,$$

where (4.15) and (4.16) are:

$$\operatorname{Lip}\big(df_y \circ L^{-1}\big) \;\leq\; t-\varepsilon \qquad \operatorname{Lip}\big(L \circ df_y^{-1}\big) \;\leq\; \big(t^{-1}+\varepsilon\big)^{-1} . \tag{4.15}$$

$$\big|f(z) - f(y) - df_y(z-y)\big| \;\leq\; \varepsilon|L(z-y)| \qquad \forall z \in B_{2/i}(y) . \tag{4.16}$$

Step 1 The sets $E(x,L,i)$ enjoy the properties (ii) and (iii).

It is not difficult to conclude (4.14) from (4.15), using elementary linear algebra. Moreover, note that (4.15) and (4.16) imply

$$t^{-1}|L(z-y)| \;\leq\; |f(z)-f(y)| \;\leq\; t|L(z-y)| \tag{4.17}$$

for every $y, z \in E(x,L,i)$. Therefore, $f|_{E_i}$ is injective and (4.13) follows easily.

Step 2 The sets $E(x,L,i)$ cover $E \setminus E_0$.

Let $y \in E \setminus E_0$. Then f is differentiable at y and df_y is injective. Therefore, from the density of S in the set of injective linear maps $A : \mathbb{R}^k \to \mathbb{R}^n$, it follows that there exists an L for which the bounds (4.15) hold.

Since L is injective, there exists a $c > 0$ such that $c|v| \leq |L(v)|$ for every $v \in \mathbb{R}^k$. From the differentiability of f at y, it follows that for some $i > 0$ we have

$$\big|f(z) - f(y) - df_y(z-y)\big| \;\leq\; \varepsilon c|z-y| \;\leq\; \varepsilon|L(z-y)| \qquad \forall z \in B_{2/i}(y) .$$

From the density of C, there exists $x \in C$ such that $y \in B_{1/i}(x)$. Therefore, $y \in E(x,L,i)$. This shows that the sets $\{E(x,L,i)\}$ cover $E \setminus E_0$ and concludes the proof. $\square$

Proof of the Area Formula. Fix $t > 1$ and let $\{E_i\}$ be the sets of Lemma 4.12. Define inductively $\tilde{E}_0 := E_0$ and

$$\tilde{E}_i \;:=\; E_i \setminus \bigcup_{j=0}^{i-1} \tilde{E}_j .$$

Then $\{\tilde{E}_j\}$ is a Borel partition of E. We claim that

$$\int_{f(E)} \mathscr{H}^0(f^{-1}(y))\, d\mathscr{H}^k(y) \;=\; \sum_{i\geq 0} \int_{f(\tilde{E}_i)} \mathscr{H}^0\big((f|_{\tilde{E}_i})^{-1}(y)\big)\, d\mathscr{H}^k(y) . \tag{4.18}$$

We will prove this equality later. First we will show how to combine it with the lemmas proved above in order to obtain (4.1).

From Lemmas 4.10 and 4.11 it follows that

$$\int_{f(\tilde{E}_0)} \mathscr{H}^0\big((f|_{\tilde{E}_0})^{-1}(y)\big)\, d\mathscr{H}^k(y) \;=\; \int_{\tilde{E}_0} Jf(x)\, d\mathscr{L}^k(x) . \tag{4.19}$$

From Lemma 4.12(ii) it follows that

$$\int_{f(\tilde{E}_i)} \mathscr{H}^0\big((f|_{\tilde{E}_i})^{-1}(y)\big)\, d\mathscr{H}^k(y) \;=\; \mathscr{H}^k\big(f(\tilde{E}_i)\big) . \tag{4.20}$$

From (4.13) and Proposition 2.12(iv) we conclude

$$t^{-n}\mathscr{H}^k\big(L_i(\tilde{E}_i)\big) \;\le\; \mathscr{H}^k\big(f(\tilde{E}_i)\big) \;\le\; t^n\mathscr{H}^k\big(L_i(\tilde{E}_i)\big)\,. \tag{4.21}$$

From Lemma 4.9 we obtain

$$\mathscr{H}^k\big(L_i(\tilde{E}_i)\big) \;=\; \int_{\tilde{E}_i} JL_i(x)\,d\mathscr{L}^k(x)\,, \tag{4.22}$$

and finally from (4.14) we have

$$t^{-n}\int_{\tilde{E}_i} Jf(x)\,d\mathscr{L}^k(x) \;\le\; \int_{\tilde{E}_i} JL_i(x)\,d\mathscr{L}^k(x) \;\le\; t^n\int_{\tilde{E}_i} Jf(x)\,d\mathscr{L}^k(x)\,. \tag{4.23}$$

From (4.20), (4.21), (4.22), and (4.23), we obtain

$$t^{-2n}\int_{\tilde{E}_i} Jf(x)\,d\mathscr{L}^k(x) \;\le\; \int_{f(\tilde{E}_i)} \mathscr{H}^0\big((f|_{\tilde{E}_i})^{-1}(y)\big)\,d\mathscr{H}^k(y) \;\le\; t^{2n}\int_{\tilde{E}_i} Jf(x)\,d\mathscr{L}^k(x)\,. \tag{4.24}$$

Therefore, from (4.18), (4.19), and (4.24) we conclude

$$t^{-2n}\int_{E} Jf(x)\,d\mathscr{L}^k(x) \;\le\; \int_{f(E)} \mathscr{H}^0(f^{-1}(y))\,d\mathscr{H}^k(y) \;\le\; t^{2n}\int_{E} Jf(x)\,d\mathscr{L}^k(x)\,. \tag{4.25}$$

Letting $t \downarrow 1$ we obtain the desired formula.

To complete the proof we need to show that (4.18) is valid. First of all, note that for any $N \in \mathbb{N}$, the following inequality is trivial

$$\int_{f(E)} \mathscr{H}^0(f^{-1}(y))\,d\mathscr{H}^k(y) \;\ge\; \sum_{i=0}^{N}\int_{f(\tilde{E}_i)} \mathscr{H}^0\big((f|_{\tilde{E}_i})^{-1}(y)\big)\,d\mathscr{H}^k(y)\,. \tag{4.26}$$

Letting $N \uparrow \infty$ we conclude

$$\int_{f(E)} \mathscr{H}^0(f^{-1}(y))\,d\mathscr{H}^k(y) \;\ge\; \sum_{i\ge 0}\int_{f(\tilde{E}_i)} \mathscr{H}^0\big((f|_{\tilde{E}_i})^{-1}(y)\big)\,d\mathscr{H}^k(y)\,. \tag{4.27}$$

Therefore, to prove (4.18) it suffices to show:

$$\int_{f(E)} \mathscr{H}^0(f^{-1}(y))\,d\mathscr{H}^k(y) \;\le\; \sum_{i\ge 0}\int_{f(\tilde{E}_i)} \mathscr{H}^0\big((f|_{\tilde{E}_i})^{-1}(y)\big)\,d\mathscr{H}^k(y)\,. \tag{4.28}$$

After possibly subdividing each $\tilde{E}_i$ into a countable number of subsets, we can assume that $\mathscr{L}^k(\tilde{E}_i) < \infty$. Define $F_N := \bigcup_{i=0}^N \tilde{E}_i$ and note that

$$\sum_{i\ge 1}\int_{f(\tilde{E}_i)} \mathscr{H}^0\big((f|_{\tilde{E}_i})^{-1}(y)\big)\,d\mathscr{H}^k(y) \;=\; \lim_{N\uparrow\infty}\int_{f(F_N)} \mathscr{H}^0\big(f^{-1}(y)\cap F_N\big)\,d\mathscr{H}^k(y)\,. \tag{4.29}$$

For every $N \ge M$ we can write

$$\int_{f(F_N)} \mathscr{H}^0\big(f^{-1}(y)\cap F_N\big)\,d\mathscr{H}^k(y) \;\;\ge\;\; \int \mathscr{H}^0\big(f^{-1}(y)\cap F_N\big)\,d\big[\mathscr{H}^k \llcorner f(F_M)\big](y)\,. \tag{4.30}$$

Since $\mathscr{H}^k(F_N) < \infty$ and f is Lipschitz, from Proposition 2.12(iv) we obtain $\mathscr{H}^k(f(F_M)) < \infty$. Therefore, the measure $\mu_M := \mathscr{H}^k \llcorner f(F_M)$ is finite. Moreover, the function

$$g_N(y) \;:=\; \mathscr{H}^k\big(f^{-1}(y)\cap F_N\big)$$

converges pointwise on $f(F_M)$ to $g(y) := \mathscr{H}^k(f^{-1}(y))$. Hence, letting $N \uparrow \infty$ in (4.30), we conclude

$$\sum_{i\geq 1} \int_{f(\tilde{E}_i)} \mathscr{H}^0\big((f|_{\tilde{E}_i})^{-1}(y)\big)\, d\mathscr{H}^k(y) \;\geq\; \int \mathscr{H}^0(f^{-1}(y))\, d\mu_M(y)\,. \tag{4.31}$$

Since $F_M \uparrow E$, from the σ–additivity of the Hausdorff measure we obtain (4.28). □

3. The Geometric Lemma and the Rectifiability Criterion

Proof of Lemma 4.7. The condition (4.3) implies that the map $P_V|_F$ is injective. Let $x_1, \ldots, x_k, y_1, \ldots, y_{n-k}$ be a system of orthonormal coordinates such that

$$V \;=\; \big\{(x,y) \;:\; y \;=\; 0\big\}\,.$$

It follows that for every $x \in G := P_V(F)$ there exists a unique $y \in V^\perp$ such that $(x,y) \in G$. Hence, we can define a function $g : G \to V^\perp$ such that

$$F \;=\; \big\{(x, g(x)) \;:\; x \in G\big\}\,.$$

Note that if $z_1 = (x_1, g(x_1))$ and $z_2 = (x_2, g(x_2))$, then

$$P_V\big(z_1 - z_2\big) \;=\; x_1 - x_2 \qquad \text{and} \qquad Q_V\big(z_1 - z_2\big) \;=\; g(x_1) - g(x_2)\,.$$

Therefore, (4.3) can be translated into $|g(z_1) - g(z_2)| \leq \alpha|z_1 - z_2|$ and we conclude that g is Lipschitz. Proposition 2.17 shows that there exists a Lipschitz extension of g to V. This concludes the proof. □

Proof of Proposition 4.6. We remark that all the sets defined in this proof are Borel. Checking this is a standard exercise in measure theory.

First of all we define the sets

$$F_{i,j} \;:=\; \left\{x \in E : \mathscr{H}^k(E \cap B_r(x)) \;\geq\; \frac{r^k}{j} \qquad \text{for all } r < 1/i\right\}\,, \tag{4.32}$$

where i and j are positive integers. Since $E \subset \bigcup_{i,j} F_{i,j}$, it suffices to prove that each $F_{i,j}$ is rectifiable. From now on we restrict our attention to a fixed $F_{i,j}$ and we drop the indices i, j to simplify the notation.

Next we fix a finite collection of linear planes $\{V_1, \ldots, V_N\}$ such that for every linear plane V we have

$$C(0, V, \alpha) \;\subset\; C(0, V_m, 2\alpha) \qquad \text{for some } V_m.$$

For any fixed $\varepsilon > 0$ we define the sets

$$G^\varepsilon_{l,m} \;:=\; \Big\{x \in F : \mathscr{H}^k\big(E \cap B_r(x) \setminus C(x, V_m, 2\alpha)\big) \;\leq\; \varepsilon r^k \qquad \text{for all } r < 1/l\Big\}\,.$$

Clearly, we have $F \subset \bigcup_{l,m} G^\varepsilon_{l,m}$. Therefore, it suffices to prove that for some $\varepsilon > 0$ each $G^\varepsilon_{l,m}$ is rectifiable. We will be able to show this provided that ε is smaller than a geometric constant $c(\alpha, j)$, where j is the parameter which appears in definition (4.32). Therefore, the choice of ε is independent of l and m.

We set $2\rho := \min\{j^{-1}, l^{-1}\}$ and we will prove that $G^\varepsilon_{l,m} \cap B_\rho(y)$ is a subset of a k–dimensional Lipschitz graph for every y and whenever $\varepsilon < c(\alpha, j)$. Without loss of generality, we carry out the proof for the case $G := G^\varepsilon_{l,m} \cap B_\rho(0)$.

Let us briefly summarize the properties enjoyed by G:

(a) $G \subset B_\rho(0)$;

(b) $\mathscr{H}^k(E \cap B_r(x)) \geq j^{-1} r^k$ for every $x \in G$ and every $r < 2\rho$;

(c) $\mathscr{H}^k(E \cap B_r(x) \setminus C(x, V_m, 2\alpha)) \leq \varepsilon r^k$ for every $x \in G$ and $r < 2\rho$.

We claim that there exists a constant $c(\alpha, j)$ such that

$$\text{if } \varepsilon < c(\alpha, j) \qquad \text{then} \qquad G \subset C(x, V_m, 4\alpha) \quad \text{for every } x \in G\,. \tag{4.33}$$

In view of Lemma 4.7 this claim concludes the proof.

We now come to the proof of (4.33). First of all, note that for every α there exists a constant $c(\alpha) < 1$ such that for every cone $C(z, V, 4\alpha)$ we have:

$$\text{if } y \notin C(z, V, 4\alpha), \qquad \text{then} \quad B_{c(\alpha)|y-x|}(y) \cap C(z, V, 2\alpha) = \emptyset\,; \tag{4.34}$$

cf. Figure 2.

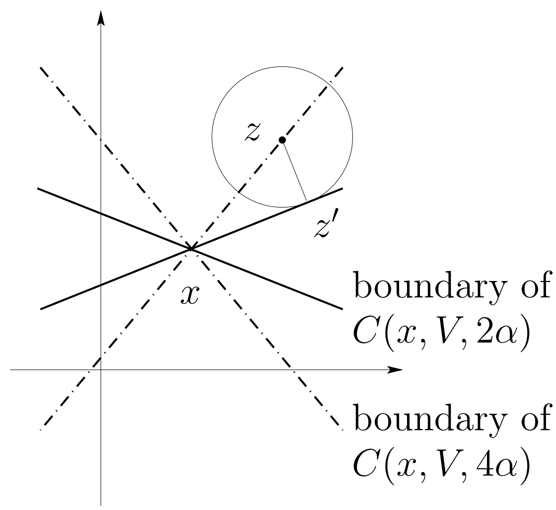

FIGURE 2. The geometric constant $c(\alpha)$ of (4.34) is given by $|z' - z|/|z - x|$, where z is any point distinct from y which belongs to the boundary of $C(x, V, 4\alpha)$.

Then (4.33) holds for

$$c(\alpha, j) \ := \ \frac{[c(\alpha)]^k}{2^k j}\,.$$

Indeed, assume by contradiction that the conclusion of (4.33) is false. Then there exist $x, y \in G$ such that $y \notin C(x, V_m, 4\alpha)$. From (a) we know that $r := |x - y| \leq \rho$. From (4.34) we obtain

$$B_{c(\alpha)r}(y) \ \subset \ \mathbb{R}^n \setminus C(x, V_m, 2\alpha)\,.$$

From (b) we conclude

$$\mathscr{H}^k\big(B_{2r}(x) \cap E \setminus C(x, V_m, 2\alpha)\big) \ \geq \ \mathscr{H}^k\big(B_{c(\alpha)r}(y) \cap E\big) \ \geq \ \frac{(c(\alpha)r)^k}{j}\,.$$

Therefore, from (c) we obtain $\varepsilon(2r)^k \geq j^{-1}(c(\alpha)r)^k$, which yields $\varepsilon \geq j^{-1}2^{-k}[c(\alpha)]^k = c(\alpha, j)$. This contradicts the choice $\varepsilon < c(\alpha, j)$ and therefore concludes the proof. $\square$

4. Proof of Theorem 4.8

Proof of Theorem 4.8. We assume, without loss of generality, that the measure μ is finite.

From tangent measures to rectifiability.

Let x be a point where

$$\mathrm{Tan}_k(\mu, x) \;=\; \left\{c_x \mathscr{H}^k \llcorner V_x\right\},$$

where c_x is a positive constant and V_x a k–dimensional linear plane. We first prove that

$$\infty \;>\; \theta^{k*}(\mu, x) \;\geq\; \theta^k_*(\mu, x) \;>\; 0\,. \tag{4.35}$$

Let $0 \leq \varphi \leq 1$ be a compactly supported continuous function such that $\varphi = 1$ on $B_{1/2}(0)$. Then

$$\lim_{r \downarrow 0} \frac{1}{r^k} \int \varphi(y)\, d\mu_{x,r}(y) \;=\; c_x \int_{V_x} \varphi(y)\, d\mathscr{H}^k(y) \;\in\; [c_x \omega_k 2^{-k}, c_x \omega_k]\,.$$

Hence, we conclude

$$\liminf_{r \downarrow 0} \frac{\mu(B_r(x))}{\omega_k r^k} \;\geq\; \lim_{r \downarrow 0} \frac{1}{\omega_k r^k} \int \varphi(y)\, d\mu_{x,r}(y) \;=\; c_x 2^{-k}\,,$$

and

$$\limsup_{r \downarrow 0} \frac{\mu(B_r(x))}{\omega_k r^k} \;\leq\; \lim_{r \downarrow 0} \frac{1}{\omega_k r^k} \int \varphi(y)\, d\mu_{x,2r}(y) \;\leq\; c_x 2^k\,.$$

Therefore, (4.35) holds and we can apply Proposition 2.16 to conclude that $\mu = f\mathscr{H}^k \llcorner E$ for some Borel function f and some Borel set E.

In order to show that μ is rectifiable, it suffices to prove that $E_c := E \cap \{f > c\}$ is rectifiable. We fix $c > 0$ and consider $\nu := \mathscr{H}^k \llcorner E_c$. From Proposition 3.12 it follows that

$$\mathrm{Tan}_k(\nu, x) \;=\; \left\{c_x \mathscr{H}^k \llcorner V_x\right\} \qquad \text{for } \mu\text{–a.e. } x, \tag{4.36}$$

where c_x is a positive constant and V_x a k–dimensional plane. We wish to apply Proposition 4.6. Arguing as above, we clearly have

$$\theta^k_*(E_c, x) \;>\; 0 \qquad \text{for } \mathscr{H}^k\text{–a.e. } x \in E_c.$$

Moreover, $\mathscr{H}^k(E_c) \leq c^{-1}\mu(\mathbb{R}^n) < \infty$. Thus, it suffices to check the condition on cones (4.2) and we will do it for all points x where (4.36) holds. Indeed, fix an $\alpha > 0$ and note that

$$\limsup_{r \downarrow 0} \frac{\mathscr{H}^k\big(E_c \cap B_r(x) \setminus C(x, V_x, \alpha)\big)}{r^k} \;=\; \limsup_{r \downarrow 0} \frac{\nu_{x,r}\big(B_1(0) \setminus C(0, V_x, \alpha)\big)}{r^k}\,. \tag{4.37}$$

Note that the set $A := B_1(0) \setminus C(0, V_x, \alpha)$ is open and bounded and $c_x \mathscr{H}^k \llcorner V_x(\partial A) = 0$. Therefore, from Proposition 2.7 we conclude

$$\limsup_{r \downarrow 0} \frac{\nu_{x,r}\big(B_1(0) \setminus C(0, V_x, \alpha)\big)}{r^k} \;=\; c_x \mathscr{H}^k\big(V_x \cap B_1(0) \setminus C(0, V_x, \alpha)\big) \;=\; 0\,. \tag{4.38}$$

Hence, (4.2) holds and we can apply Proposition 4.6 to conclude that E_c is rectifiable.

From rectifiability to tangent measures.

From Proposition 3.12 it suffices to prove the claim when $\mu = \mathscr{H}^k \llcorner E$, where E is a subset of a Lipschitz graph. Thus, we assume

$$E \;=\; \big\{(z, f(z)) \;:\; z \in G\big\}\,,$$

where $G \subset \mathbb{R}^k$ is a Borel set and $f : \mathbb{R}^k \to \mathbb{R}^{n-k}$ is a Lipschitz map.

Let $H \subset G$ be the set of points z where

- f is differentiable;
- df is Lebesgue continuous;
- G has density 1 (with respect to $\mathscr{L}^k$).

Denote by V_z the k–dimensional linear plane:

$$V_z \;:=\; \big\{(y, df_z(y)) \;:\; y \in \mathbb{R}^k\big\}\,.$$

We claim that for every $z_0 \in H$ we have

$$\mathrm{Tan}_k\big(\mu, (z_0, f(z_0))\big) \;=\; \big\{\mathscr{H}^k \llcorner V_{z_0}\big\}\,.$$

Clearly, since $\mathscr{L}^k(G \setminus H) = 0$, this claim would conclude the proof of the proposition

We now come to the proof of the claim. Without loss of generality we can assume that $z_0 = 0$ and $f(z_0) = 0$. To simplify the notation, we will write V in place of V_{z_0} and we denote by F the Lipschitz map

$$F \;:\; \mathbb{R}^k \ni z \;\to\; (z, f(z)) \in \mathbb{R}^n\,.$$

Let us fix a test function $\varphi \in C_c(\mathbb{R}^m)$ and recall that

$$\frac{1}{r^k}\int \varphi(x)\, d\mu_{0,r}(x) \;=\; \frac{1}{r^k}\int \varphi\left(\frac{x}{r}\right)\, d\mu(x)\,. \tag{4.39}$$

We now use the Area Formula to write

$$\int \varphi\left(\frac{x}{r}\right)\, d\mu(x) \;=\; \int_G \varphi\left(\frac{F(z)}{r}\right) JF(z)\, d\mathscr{L}^k(z)\,. \tag{4.40}$$

Let $C > 0$ be such that $\varphi \in C_c(B_C(0))$. Then we have

$$\int_G \varphi\left(\frac{F(z)}{r}\right) JF(z)\, d\mathscr{L}^k(z) \;=\; \int_{G \cap B_{Cr/(\mathrm{Lip}\,F)}(0)} \varphi\left(\frac{F(z)}{r}\right) JF(z)\, d\mathscr{L}^k(z)\,. \tag{4.41}$$

Recall that

- 0 is a point of density 1 for G and therefore $r^{-k}\mathscr{L}^k(B_{Cr}(0) \setminus G)$ vanishes;
- dF is Lebesgue continuous at 0, and therefore

$$\lim_{r \downarrow 0} r^{-k} \int_{B_{Cr}(0)} |JF(z) - JF(0)|\, d\mathscr{L}^k(z) \;=\; 0\,;$$

- F is differentiable at 0 and hence

$$\lim_{r \downarrow 0} \sup_{z \in B_{Cr}(0)} \left|\varphi\left(\frac{F(z)}{r}\right) - \varphi\left(\frac{dF_0(z)}{r}\right)\right| \;=\; 0\,.$$

From these three remarks we conclude that

$$\lim_{r\downarrow 0}\frac{1}{r^k}\left\{\int_{G\cap B_{Cr}(0)}\varphi\left(\frac{F(z)}{r}\right)JF(z)\,d\mathscr{L}^k(z)\;-\;\int_{B_{Cr}(0)}\varphi\left(\frac{dF_0(z)}{r}\right)JF(0)\,d\mathscr{L}^k(z)\right\}\;=\;0\,. \tag{4.42}$$

Since dF_0 is linear, we have $r^{-1}dF_0(z) = dF_0(r^{-1}z)$. We then change variables to obtain

$$\begin{aligned}\frac{1}{r^k}\int_{B_{Cr}(0)}\varphi\left(\frac{dF_0(z)}{r}\right)JF(0)\,d\mathscr{L}^k(z) \;&=\; \int_{\mathbb{R}^k}\varphi(dF_0(w))JdF_0\,d\mathscr{L}^k(w)\\ &=\; \int_V \varphi(x)\,d\mathscr{H}^k(x)\,.\end{aligned} \tag{4.43}$$

Therefore, putting (4.39), (4.40), (4.41), and (4.42) together, we conclude

$$\lim_{r\downarrow 0}\frac{1}{r^k}\int\varphi(x)\,d\mu_{0,r}(x)\;=\;\int\varphi(x)\,d[\mathscr{H}^k\llcorner V](x)\,.$$

The arbitrariness of φ yields $r^{-k}\mu_{0,r}\overset{*}{\rightharpoonup}\mathscr{H}^k\llcorner V$ and completes the proof. $\square$

CHAPTER 5

The Marstrand–Mattila Rectifiability Criterion

In this chapter we will improve upon the characterization of rectifiable sets given in the previous chapter. Our goal is the following result:

THEOREM 5.1. *Let μ be a measure such that for μ–a.e. x we have*

(i) $\infty > \theta^{*k}(\mu, x) \geq \theta^k_*(\mu, x) > 0$;
(ii) *Every tangent measure to μ at x is of the form $c\mathscr{H}^k \llcorner V$ for some k–dimensional linear plane V.*

Then μ is a rectifiable measure.

Clearly, from Theorem 4.8 it follows that every rectifiable measure enjoys the property above. However, the converse is much more subtle than Theorem 4.8. Indeed there is a major difference between (ii) and (4.4): The latter implies uniqueness of the tangent measure, whereas the former does not. Indeed, there can be a point x where (i) and (ii) hold and $\mathrm{Tan}_k(\mu, x)$ consists of more than one measure. This might happen because both the constant c and the plane V of (ii) might vary, as is seen in the examples below. The case of Example 5.2 — where V varies — is more relevant, since it implies that we cannot conclude Theorem 5.1 directly from Proposition 4.6.

Note that Theorem 5.1 and Theorem 4.8 imply that, if μ enjoys the properties (i) and (ii) at μ–a.e. x, then μ has a unique tangent measure at almost every point. Therefore, the set of exceptional points where (i) and (ii) hold but the tangent measures are not unique is a set of measure zero.

EXAMPLE 5.2. *Let $\Gamma \subset \mathbb{R}^2$ be the graph of the function $f : \mathbb{R} \to \mathbb{R}$ given by*

$$f(z) := \begin{cases} |z| \sin\big(\log\big|\log\big(1+|z|^{-1}\big)\big|\big) & \text{for } z \neq 0 \\ 0 & \text{for } z = 0. \end{cases}$$

The measure $\mu := \mathscr{H}^1 \llcorner \Gamma$ is locally finite and satisfies both conditions (i) and (ii) of Theorem 5.1 at every $x \in \Gamma$. If we denote by ℓ_a the line $\ell_a := \big\{(z, az) : z \in \mathbb{R}\big\}$, then

$$\mathrm{Tan}_1(\mu, 0) = \big\{\mathscr{H}^1 \llcorner \ell_a \ : \ a \in [-1, 1]\big\}.$$

EXAMPLE 5.3. *Similarly, we let $g : \mathbb{R}^2 \to [1, 3]$ be given by*

$$g(x_1, x_2) = 2 + \sin\big(\log\big|\log(1+|x_1|^{-1})\big|\big)$$

(actually g is not defined on $\{x_1 = 0\}$ but this does not affect the discussion). Then the measure of $\mathbb{R}^2$ given by $\mu = g\,\mathscr{H}^1 \llcorner \ell_0$ satisfies both (i) and (ii) at every $x \in \ell_0$. However

$$\mathrm{Tan}_1(\mu, 0) = \big\{c\mathscr{H}^1 \llcorner \ell_0 \ : \ c \in [1, 3]\big\}.$$

Weakly linearly approximable sets. Theorem 5.1 is a corollary of a more general rectifiability criterion for k–dimensional sets of $\mathbb{R}^n$, first proved by Marstrand for $k = 2$ and $n = 3$ in [**16**] and later generalized by Mattila in [**18**].

DEFINITION 5.4. *Let E be a k–dimensional set of $\mathbb{R}^n$ and fix $x \in \mathbb{R}^n$. We say that E is* weakly linearly approximable *at x if for every $\eta > 0$ there exists λ and r positive such that*

- *For every $\rho < r$ there exists a k–dimensional linear plane W (which possibly depend on ρ) for which the following two conditions hold:*

$$\mathscr{H}^k\big(E \cap B_\rho(x) \setminus \big\{z : \operatorname{dist}(x+W, z) \le \eta\rho\big\}\big) \;<\; \eta\rho^k\,; \tag{5.1}$$

$$\mathscr{H}^k\big(E \cap B_{\eta\rho}(z)\big) \;\ge\; \lambda\rho^k \qquad \text{for all } z \in (x+W) \cap B_\rho(x). \tag{5.2}$$

The first condition tells us that, at small scales around x, most of E is contained in a tubular neighborhood of $x + W$; see Figure 1.

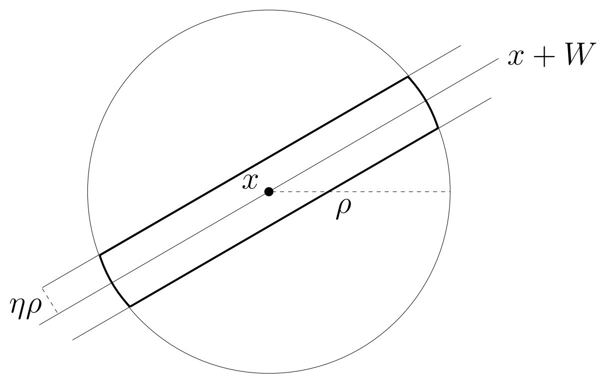

FIGURE 1. The set given by the intersection of the ball $B_\rho(x)$ with the strip $\big\{z : \operatorname{dist}(x+W, z) \le \eta\rho\big\}$. Condition (5.1) implies that most of $E \cap B_\rho(x)$ lies within this set.

The second condition says that at small scales, any small ball centered around a point z of $x + W$ contains a significant portion of E; see Figure 2.

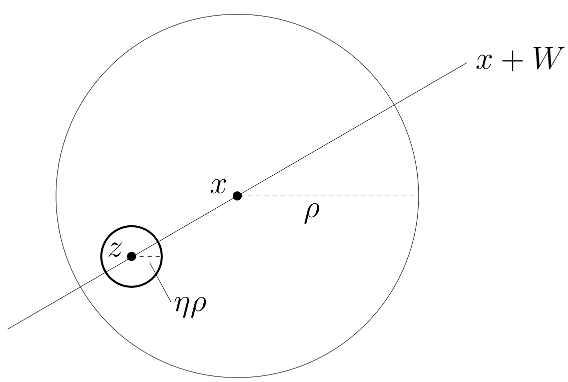

FIGURE 2. A point z on $(x + W) \cap B_\rho(x)$ and the small ball $B_{\eta\rho}(z)$ centered on it. According to (5.2) this ball contains a significant portion of E.

If $\mu := f\mathscr{H}^k \llcorner E$ is as in Theorem 5.1, then these two conditions are satisfied at $\mathscr{H}^k$–almost every point of E. Therefore, Theorem 5.1 follows from the following proposition.

Proposition 5.5 (Marstrand–Mattila Rectifiability Criterion). *Let E be a Borel set such that $0 < \mathscr{H}^k(E) < \infty$ and assume that E is weakly linearly approximable at $\mathscr{H}^k$–a.e. $x \in E$. Then E is rectifiable.*

Plan of the chapter. In the first section we introduce some preliminary definitions and lemmas and in the second we prove Proposition 5.5. In the final section we show how Theorem 5.1 follows from Proposition 5.5.

1. Preliminaries: Purely unrectifiable sets and projections

First we introduce the definition of purely unrectifiable set.

Definition 5.6. *Let E be a k–dimensional set with $\mathscr{H}^k(E) > 0$. We say that E is* purely unrectifiable *if for every Lipschitz k–dimensional graph Γ we have $\mathscr{H}^k(\Gamma \cap E) = 0$.*

The following decomposition property of Borel sets with finite Hausdorff measure is a simple corollary of our definition:

Theorem 5.7 (Decomposition Theorem). *Let E be a Borel set such that $\mathscr{H}^k(E) < \infty$. Then there exist two Borel sets $E^u, E^r \subset E$ such that*

- *$E^u \cup E^r = E$;*
- *E^r is rectifiable;*
- *E^u is purely unrectifiable.*

Such a decomposition is unique up to $\mathscr{H}^k$–null sets, that is: If F^u and F^r satisfy the three properties listed above, then

$$\mathscr{H}^k(E^r \setminus F^r) \;=\; \mathscr{H}^k(F^r \setminus E^r) \;=\; \mathscr{H}^k(E^u \setminus F^u) \;=\; \mathscr{H}^k(F^u \setminus E^u) \;=\; 0\,. \tag{5.3}$$

Proof. We define

$$\mathcal{R}(E) \;:=\; \big\{E' \;:\quad E' \subset E \text{ is Borel and rectifiable}\big\}\,.$$

and

$$\alpha \;:=\; \sup_{E' \in \mathcal{R}(E)} \mathscr{H}^k(E')\,.$$

Let $\{E_i\} \subset \mathcal{R}(E)$ be such that $\mathscr{H}^k(E_i) \uparrow \alpha$. Then we set $E^r := \bigcup E_i$. Clearly E^r is rectifiable, $E^r \subset E$, and $\mathscr{H}^k(E^r) = \alpha$. We claim that $E^c := E \setminus E^r$ is purely unrectifiable. Indeed, if there were a Lipschitz graph Γ such that $\mathscr{H}^k(E^c \cap \Gamma) > 0$ we would have that

$$\mathscr{H}^k\big(E^r \cup (\Gamma \cap E^c)\big) \;>\; \alpha\,. \tag{5.4}$$

Since $E^r \cup (\Gamma \cap E^c) \in \mathcal{R}(E)$, (5.4) would contradict the maximality of α.

To prove the uniqueness of the decomposition, note that the intersection between a purely unrectifiable set and a rectifiable set always has $\mathscr{H}^k$ measure 0. Therefore, if F^r and F^u are as in the statement of the theorem we have

$$\mathscr{H}^k(E^r \cap E^u) \;=\; \mathscr{H}^k(E^r \cap F^u) \;=\; \mathscr{H}^k(F^r \cap E^u) \;=\; \mathscr{H}^k(F^r \cap F^u) \;=\; 0\,. \tag{5.5}$$

Since $E^r \cup E^u = E = F^r \cup F^u$, (5.5) implies (5.3). $\square$

We are now ready for the following two lemmas. The first is a trivial application of the Decomposition Theorem and of the Besicovitch Differentiation Theorem. The second relies upon the Geometric Lemma of the previous chapter. Note that both of them will no longer be required after proving Proposition 5.5.

LEMMA 5.8. *If Proposition 5.5 were false, there would exist a purely unrectifiable set E which is weakly linearly approximable at $\mathscr{H}^k$–a.e. $x \in E$.*

PROOF. Assume that Proposition 5.5 is false and let F be an unrectifiable set which is weakly linearly approximable at $\mathscr{H}^k$–a.e. $x \in F$. Let $F^r \cup F^u$ be the decomposition of F into a rectifiable part and purely unrectifiable part given by Lemma 5.8. Recalling Proposition 2.2, we have that

$$\lim_{r\downarrow 0} \frac{\mathscr{H}^k(F^u \cap B_r(x))}{\mathscr{H}^k(F \cap B_r(x))} \;=\; 1 \tag{5.6}$$

for $\mathscr{H}^k$–a.e. $x \in F^u$. Moreover, note that, if F is weakly linearly approximable at x and x satisfies (5.6), then F^u is also weakly linearly approximable at x.

Therefore, we conclude that F^u is purely unrectifiable and weakly linearly approximable at $\mathscr{H}^k$–a.e. $x \in F^u$. □

LEMMA 5.9. *Let E be a purely unrectifiable set with finite Hausdorff measure and which is weakly linearly approximable at $\mathscr{H}^k$–a.e. $x \in E$. Then $\mathscr{H}^k(P_V(E)) = 0$ for* every *k–dimensional linear plane V.*

REMARK 5.10. *From the Besicovitch–Federer Projection Theorem (see for instance Theorem 18.1 of* [**21**]*) we know that every purely unrectifiable set E with finite $\mathscr{H}^k$ measure has null projection on* almost every *k–dimensional plane V. Here, "almost every" is with respect to the natural measure that one can define on the set of k–dimensional linear planes of $\mathbb{R}^m$ (the so called Grassmannian manifold $G(m,k)$); cf. Section 3.9 of* [**21**]*. However, one can find examples of purely unrectifiable sets which project to sets with positive measure on* some *k–dimensional plane; see for instance Lemma 18.12 of* [**21**].

PROOF. We fix $0 < \varepsilon < 1/2$.

Step 1 As is often the case, we start by selecting a compact set $C \subset E$ such that $\mathscr{H}^k(E \setminus C) < \varepsilon$ and the conditions of weak linear approximation of C hold at every point $x \in C$ in a uniform way. More precisely

(Cl) There exists a compact set $C \subset E$ and positive numbers r_0, η, δ such that

$$\mathscr{H}^k(E \setminus C) \;<\; \varepsilon \qquad \eta \;<\; \delta\varepsilon \;<\; \varepsilon \tag{5.7}$$

and for every $a \in C$ and every $r < r_0$ the following two properties hold:

$$\mathscr{H}^k(E \cap B_r(a)) \;\geq\; \delta r^k\,, \tag{5.8}$$

$$\text{there exists a } k\text{–plane } W \text{ s.t. } C \cap B_r(a) \;\subset\; \big\{z \;:\; \operatorname{dist}(z, a+W) \;\leq\; \eta r\big\}\,. \tag{5.9}$$

In order to show (Cl) we first select C' compact such that:

- $\mathscr{H}^k(E \setminus C') < \varepsilon/2$;
- There are positive r_1 and δ such that condition (5.8) holds for every $r < r_1$.

This is clearly possible since (5.2) implies that the lower density of E is positive at $\mathscr{H}^k$ almost every x.

Recall that E is weakly linearly approximable at $\mathscr{H}^k$–a.e. $x \in C$. Therefore, we can select:

(a) A compact set $C \subset C'$ with $\mathscr{H}^k(C' \setminus C) < \varepsilon/2$,
(b) a positive number $\eta < \delta\varepsilon$, and
(c) a positive $r_0 < r_1$,

such that for every $a \in C$ and every $r < r_0$ there exists W satisfying

$$\mathscr{H}^k\big(E \cap B_{2r}(a) \setminus \big\{z \;:\; \mathrm{dist}\,(z, a+W) \le \eta r/2\big\}\big) \;<\; \delta\left(\frac{\eta r}{2}\right)^k . \tag{5.10}$$

Clearly, C, δ, η and r_0 satisfy both (5.7) and (5.8). We claim, that for $a \in C$ and $r < r_0$, the plane W of (5.10) also meets condition (5.9). Let us begin by assuming the contrary. Then there would exist $z \in C \cap B_r(a)$ with $\mathrm{dist}\,(z, a+W) > \eta r$. Therefore, $B_{\eta r/2}(z)$ would be contained in $B_{2r}(a) \setminus \big\{z \;:\; \mathrm{dist}\,(z, a+W) \le \eta r/2\big\}$. Hence, we would have

$$\mathscr{H}^k\big(B_{2r}(a) \cap E \setminus \big\{z \;:\; \mathrm{dist}\,(z, a+W) \le \eta r/2\big\}\big) \;\ge\; \mathscr{H}^k\big(E \cap B_{\eta r/2}(a)\big) \;\stackrel{(5.8)}{\ge}\; \delta\left(\frac{\eta r}{2}\right)^k ,$$

which contradicts (5.10).

Step 2 Now let us fix an arbitrary k–dimensional linear plane V. For each $i \in \mathbb{N}$ define the sets

$$C_i \;:=\; \big\{a \in C \;:\; C \cap B_{i^{-1}}(a) \setminus C(a, V, \eta^{-1}) = \emptyset\big\} .$$

By the Geometric Lemma 4.7, the intersection of C_i with a ball of radius i^{-1} is contained in a Lipschitz graph. Therefore, since C is purely unrectifiable, we have

$$\mathscr{H}^k\left(\bigcup_i C_i\right) \;=\; 0 .$$

It follows that for $\mathscr{H}^k$–a.e. $a \in C$ there exists $b \in C \cap B_{r_0}(a) \cap B_{i^{-1}}(a)$ such that

$$\frac{|Q_V(b-a)|}{\eta} \;>\; |P_V(b-a)|$$

and hence

$$|P_V(b-a)| \;<\; \eta|b-a| .$$

Set $r := |a-b|$, let W satisfy (5.9) and define $c := P_W(b-a) + a$. From the first step it follows that $|c-b| \le \eta r$. Since P_W is a projection, we obtain $|c-a| \le |b-a| = r$. Moreover, recalling that $\eta < \varepsilon < 1/2$, we obtain

$$|c-a| \;\ge\; |b-a| - |c-b| \;\ge\; (1-\eta)r \;>\; r/2 .$$

Therefore, the vector $w := (c-a)/|c-a|$ is a unit vector which belongs to W and is such that $|P_V(w)| \le C\eta$, where C is a geometric constant, independent of η. We claim that this implies

$$\mathscr{H}^k\left(P_V\left(\big\{z \;:\; \mathrm{dist}\,(z, a+W) < \eta r\big\} \cap B_r(a)\right)\right) \;\le\; C_1\eta r^k , \tag{5.11}$$

where C_1 is a geometric constant; cf. Figure 3.

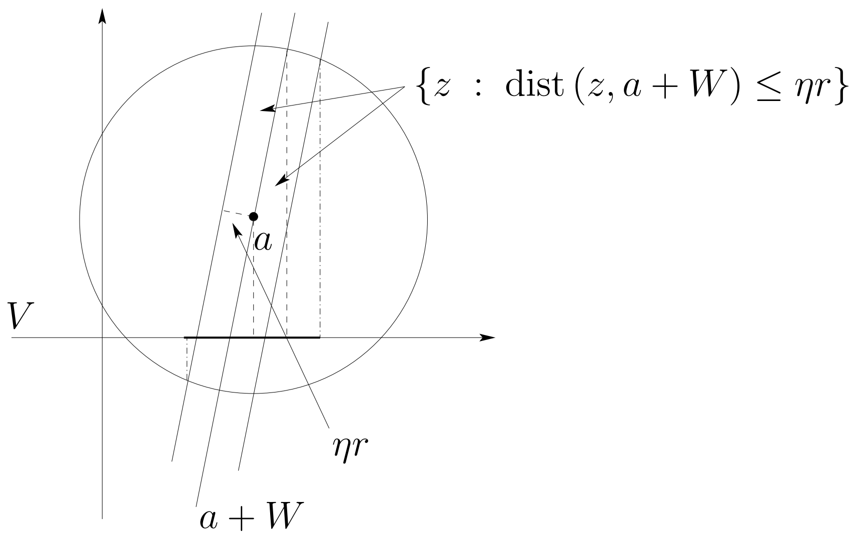

FIGURE 3. The projection of the set $\big\{z \ : \ \mathrm{dist}\,(z, a+W) < \eta r\big\} \cap B_r(a)$ has size comparable to η along at least one direction.

Indeed, after translating and rescaling, (5.11) is equivalent to showing that

$$\mathscr{H}^k\Big(P_V\Big(\big\{z \ : \ |Q_W(z)| < \eta\big\} \cap B_1(0)\Big)\Big) \ \leq \ C_1 \eta \,. \tag{5.12}$$

Now, let W' be the subspace of W perpendicular to w and set $V' := P_V(W')$. V is a linear space and its dimension is at most $m-1$. Hence, there exists a vector $v \in V$ perpendicular to V' with $|v| = 1$. Clearly, $|\langle v, w\rangle| \leq |P_V(w)| \leq \eta$. We conclude that $|\langle \zeta, v\rangle| \leq \eta$ for every $\zeta \in W \cap B_1(0)$. Therefore, for every $\zeta \in B_1(0)$ we can compute

$$|\langle \zeta, v\rangle| \ \leq \ |\langle P_W(\zeta), v\rangle| + |\langle Q_W(\zeta), v\rangle| \ \leq \ \eta + |Q_W(\zeta)| \,.$$

This equation implies that

$$P_V\Big(\big\{z \ : \ |Q_W(z)| < \eta\big\} \cap B_1(0)\Big) \ \subset \ \{z \ : \ |\langle z, v\rangle| \leq 2\eta\} \cap B_1(0)$$

and since $v \in V$ and $|v| = 1$, this establishes (5.12) and hence (5.11).

Combining (5.11) with (5.9) we obtain

$$\mathscr{H}^k\big(P_V(C \cap B_r(a))\big) \ \leq \ C_1 \eta r^k$$

and hence

$$\mathscr{H}^k\big(P_V(C \cap \overline{B}_{r/2}(a))\big) \ \leq \ C_1 \eta r^k \,. \tag{5.13}$$

Step 3 Using a Vitali–Besicovitch covering we can cover $\mathscr{H}^k$–almost all C with balls $\overline{B}_{r_i}(a_i)$ which are pairwise disjoint, centered at points of C and with radii less than $r_0/2$.

Hence, we can write

$$\begin{aligned}
\mathscr{H}^k\big(P_V(C)\big) &\leq \sum_i \mathscr{H}^k\big(P_V(C\cap \overline{B}_{r_i}(a_i))\big) \overset{(5.13)}{\leq} \sum_i C_1\eta r_i^k \\
&\overset{(5.8)}{\leq} C_1\big(\eta\delta^{-1}\big)\sum_i \mathscr{H}^k\big(E\cap \overline{B}_{r_i}(a_i)\big) \\
&= C_1\big(\eta\delta^{-1}\big)\mathscr{H}^k(E) \leq C_1\varepsilon\mathscr{H}^k(E)\,.
\end{aligned} \tag{5.14}$$

Moreover, since P_V is a projection,

$$\mathscr{H}^k\big(P_V(E\setminus C)\big) \leq \mathscr{H}^k(E\setminus C) \leq \varepsilon\,.$$

We conclude that

$$\mathscr{H}^k(P_V(E)) \leq \mathscr{H}^k\big(P_V(E\setminus C)\big)+\mathscr{H}^k\big(P_V(C)\big) \leq \big(1+C_1\mathscr{H}^k(E)\big)\varepsilon\,.$$

The arbitrariness of ε gives $\mathscr{H}^k(P_V(E))=0$, which is the desired claim. ☐

2. The proof of the Marstrand–Mattila rectifiability criterion

We argue by contradiction and assume that the proposition is false: Therefore, Lemma 5.8 gives us a purely unrectifiable set E which is weakly linearly approximable at $\mathscr{H}^k$–a.e. $x\in E$. Lemma 5.9 implies that the projection of this set on every k–dimensional plane is a null set. Then the strategy goes roughly as follows:

- We fix a ball $B_r(x)$ around a point x where E is well approximated by a plane W.
- We show that for most of the points y in $B_r(x)$ the set $E\cap B_r(y)$ is well approximated by a plane W_y which is "almost perpendicular" to W. This will follow from $\mathscr{H}^k(P_W(E\cap B_r(x)))=0$.
- The conditions of good approximation and the fact that W_y is almost perpendicular to W imply that close to y there is a "column" of pairwise disjoint balls of size smaller than (but still comparable to) r and centered at points of E. This column is almost perpendicular to W. On the other hand there must be many such points close to all the points of $W\cap B_r(x)$. Therefore, there are many of these columns of balls. See Fig. 4
- By condition (5.2), each of the balls above gives a significant contribution to $\mathscr{H}^k(E\cap B_r(x))$, which therefore turns out to be large.
- Since the upper density of a set E is bounded from above by 1 in a.e. point, the previous conclusion would give a contradiction.

Proof. We argue by contradiction and assume that the Proposition is false. From Lemma 5.8 and Lemma 5.9 we conclude the existence of a set E such that

(a) $0<\mathscr{H}^k(E)<\infty$;
(b) $\mathscr{H}^k(P_V(E))=0$ for every k–dimensional plane V;

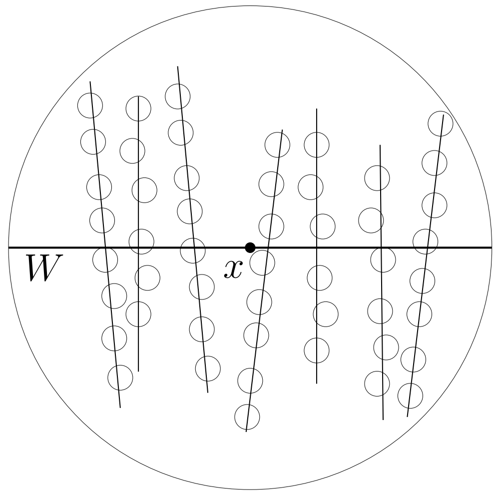

FIGURE 4. The columns of balls close to the planes W_y.

(c) E is weakly linearly approximable at $\mathscr{H}^k$–a.e. x.

Step 1 As is often the case, by standard measure theoretic arguments, we pass to a subset F which enjoys properties (a) and (b) and a strengthened version of (c). First of all we start by choosing a compact set $F \subset E$ such that

(a1) $0 < \mathscr{H}^k(F) < \infty$;

(d) There exists r_0 and δ positive such that

$$\mathscr{H}^k(E \cap B_r(a)) \;\geq\; \delta r^k \qquad \text{for all } a \in F \text{ and } r < r_0. \tag{5.15}$$

Next we fix any positive $\eta < 1$ and we claim the existence of a compact set $F_1 \subset F$ such that

(a2) $0 < \mathscr{H}^k(F_1) < \infty$;

(e) There exist $r_1 \in]0, r_0[$ and $\gamma > 0$ such that for every $r < r_1$ and every $a \in C$, we can choose a plane W with the following properties

$$F \cap B_{2r}(a) \;\subset\; \big\{z \;:\; \operatorname{dist}(z, a + W) < \eta r\big\}\,, \tag{5.16}$$

$$\mathscr{H}^k\big(E \cap B_{\eta r}(b)\big) \;\geq\; \gamma(\eta r)^k \qquad \text{for all } b \in a + W. \tag{5.17}$$

Indeed, from the definition of weak linear approximability, there exists a compact $F_1 \subset F$, $r_1 \in]0, r_0[$ and $\gamma > 0$ such that:

- (a2) holds;
- For every $r < r_1$ there exists a plane W which satisfies (5.17) and

$$\mathscr{H}^k\big(E \cap B_{2r}(a) \setminus \big\{z \;:\; \operatorname{dist}(z, a + W) \leq \eta r/2\big\}\big) \;<\; \delta\left(\frac{\eta r}{2}\right)^k . \tag{5.18}$$

Since $F_1 \subset F$, from (5.15) and (5.18) we conclude (5.16), arguing as in Step 1 of the proof of Lemma 5.9 (cf. the proof of (Cl)).

Finally, we claim the existence of $G \subset F_1$ such that

(a3) $0 < \mathscr{H}^k(G) < \infty$;

(f) There exists a positive $r_2 < r_1$ such that for every $r < r_2$ and every $a \in G$ we can choose a k–dimensional linear plane W which satisfies (5.17) and

$$F \cap B_{2r}(a) \subset \big\{z : \operatorname{dist}(z, a+W) < \eta r\big\} \tag{5.19}$$

$$(a+W) \cap B_r(a) \subset \big\{z : \operatorname{dist}(z, F) < \eta r\big\}. \tag{5.20}$$

Indeed, using the Besicovitch Differentiation Theorem and (5.15), we can select a compact set $G \subset F_1$ and a positive $r_2 < r_1$ such that (a3) holds and

$$\mathscr{H}^k\big((E \setminus F) \cap B_{2r}(a)\big) \le \gamma \left(\frac{\eta r}{2}\right)^k \qquad \forall r < r_2. \tag{5.21}$$

Now, for every $a \in G$ and $r < r_2$, select W such that (5.16) and (5.17) hold. Clearly (5.19) follows from (5.16). It remains to show that (5.20) holds. If it were false, there would be a $b \in (a+W) \cap B_r(a)$ such that $B_{\eta r}(b) \cap F = \emptyset$. Therefore,

$$\mathscr{H}^k\big(E \cap B_{\eta r}(b)\big) = \mathscr{H}^k\big((E \setminus F) \cap B_{\eta r}(b)\big) \le \mathscr{H}^k\big((E \setminus F) \cap B_{2r}(a)\big) \overset{(5.21)}{\le} \gamma \left(\frac{\eta r}{2}\right)^k,$$

which would contradict (5.17).

Step 2 We fix a $0 < t < \gamma \eta^k/2$, which will be chosen appropriately later (together with the η of the previous step).

Next we take a point $a \in G$ such that

- $\theta^{*k}(a, G) \le 1$
- $\lim_r r^{-k} \mathscr{H}^k((E \setminus G) \cap B_r(a)) = 0$.

Without loss of generality, we assume that $a = 0$ and we select $r_3 < r_2$ such that

$$\mathscr{H}^k\big(E \cap B_r(0)\big) < 2\omega_k r^k, \qquad \forall r < r_3 \tag{5.22}$$

$$\mathscr{H}^k\big((E \setminus G) \cap B_{2r}(0)\big) < t r^k \qquad \forall r < r_3. \tag{5.23}$$

After fixing $r = \sigma < r_3$, we select a W which satisfies (5.19) and (5.20). Recall that

$$\mathscr{H}^k(P_W(G)) \le \mathscr{H}^k(P_W(E)) = 0. \tag{5.24}$$

We will show that for η and t sufficiently small, (5.17), (5.19), (5.20), (5.22), (5.23), and (5.24) lead to a contradiction.

We start by introducing some notation: For $b \in W$ and $\rho \in \mathbb{R}^+$ we denote by $D_\rho(b)$ and by $C_\rho(b)$ the sets

$$D_\rho(b) := B_\rho(b) \cap W \qquad C_\rho(b) := \big\{x : P_V(x) \in D_\rho(b)\big\},$$

which we will call (respectively) the disk and cylinder centered at b of radius ρ. Their geometric meaning is illustrated in Figure 5.

We set $H := D_\sigma(0) \setminus P_W(G \cap B_\sigma(0))$. Note that H is an open set, since G is compact. For every $x \in H$ we set

$$\rho(x) := \operatorname{dist}(x, P_W(G \cap B_\sigma(0))).$$

Observe that

$$\rho(x) \le \eta\sigma. \tag{5.25}$$

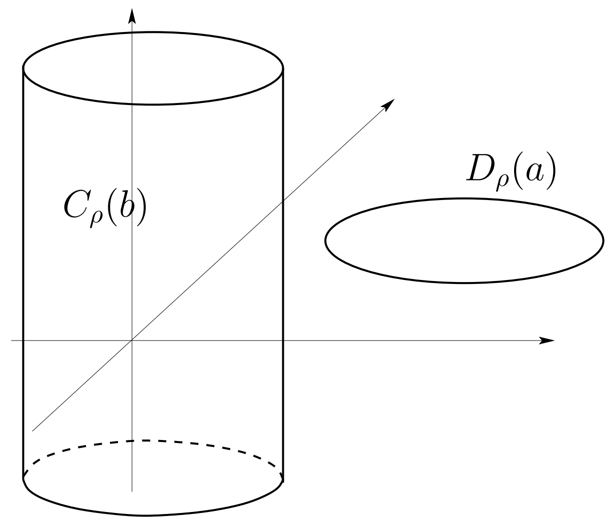

FIGURE 5. The disk $D_\rho(a)$ and the cylinder $C_\rho(b)$.

Indeed, if this were false, we would have $B_{\eta\sigma}(x) \cap G = \emptyset$. Therefore, we would conclude

$$\begin{aligned} \mathscr{H}^k\big(E \cap B_{\eta\sigma}(x)\big) &= \mathscr{H}^k\big((E \setminus G) \cap B_{\eta\sigma}(x)\big) \\ &\leq \mathscr{H}^k\big((E \setminus G) \cap B_{2\sigma}(0)\big) \overset{(5.23)}{\leq} \frac{\gamma(\eta\sigma)^k}{2}\,, \end{aligned}$$

which contradicts (5.17).

Using the $5r$–Covering Lemma, we can find a countable set $\{x_i\}_{i\in I} \subset H \cap D_{\sigma/4}(0)$ such that, if we set $\rho_i := \rho(x_i)$, we find

- The disks $\{D_{20\rho_i}(x_i)\}$ cover $H \cap D_{\sigma/4}(0)$;
- The disks $\{D_{4\rho_i}(x_i)\}$ are pairwise disjoint.

Since $\mathscr{H}^k(H \cap D_{\sigma/4}(0)) = \mathscr{H}^k(D_{\sigma/4}(0)) = \omega_k(\sigma/4)^k$, we conclude that

$$\sum_{i\in I} \omega_k \rho_i^k = \frac{1}{20^k}\sum_{i\in I} \omega_k (20\rho_i)^k \geq \frac{\mathscr{H}^k(H \cap D_{\sigma/4}(0))}{20^k} = \frac{\omega_k \sigma^k}{80^k}\,. \tag{5.26}$$

We define the subsets of indices K and J as

$$J := \big\{i \in I \,:\, C_{\rho_i/2} \cap F \cap B_\sigma(0) \neq \emptyset\big\} \qquad K := I \setminus J\,. \tag{5.27}$$

For every $i \in J$ we denote by y_i a point of $F \cap C_{\rho_i/2}$ and recall (5.19), which implies $|y_i - P_W(y_i)| \leq \eta\sigma$. Therefore, when η is chosen sufficiently small, we obtain that

$$B_{\rho_i/2}(y_i) \subset B_\sigma(0)\,.$$

Recall that $y_i \in E$ satisfies the lower density condition (5.8). Therefore, we have

$$\mathscr{H}^k\big(C_{\rho_i}(x_i) \cap (E \setminus G) \cap B_\sigma(0)\big) \geq \mathscr{H}^k\big(E \cap B_{\rho_i/2}(y_i)\big) \geq \frac{\delta\rho_i^k}{2^k}\,.$$

Hence

$$\sum_{i\in J}\omega_k\rho_i^k \;\le\; \sum_{i\in J}\frac{\omega_k 2^k}{\delta}\mathscr{H}^k\big(C_{\rho_i}(x_i)\cap(E\setminus G)\cap B_\sigma(0)\big)$$
$$\le\; \frac{\omega_k 2^k}{\delta}\mathscr{H}^k\big((E\setminus G)\cap B_\sigma(0)\big) \;\le\; \frac{t\omega_k 2^k\sigma^k}{\delta}\,,$$

where the second inequality follows because the cylinders $C_{\rho_i}(x_i)$ are pairwise disjoint. Combining this estimate with that of (5.26), we conclude that there is a positive constant c (which does not depend on *any* of the quantities r_j, η, δ, γ) such that

$$\sum_{i\in K}\omega_k\rho_i^k \;\ge\; c\sigma^k\,, \tag{5.28}$$

provided t is chosen sufficiently small.

Step 3 To simplify the notation, from now on we will write C_i in place of $C_{\rho_i}(x_i)$. For every $i\in K$, denote by z_i a point of $\partial C_{\rho_i}\cap G\cap B_\sigma(0)$. Recall that such a point exists because, according to our definition of ρ_i, we have

$$\rho_i \;:=\; \mathrm{dist}\,(x, P_W(G\cap B_\sigma(0))\,.$$

Next, we fix a k–dimensional plane W_i which meets the conditions (5.19) and (5.20) for the choice $a=z_i$, $r=\rho_i/8\eta$.

Since $i\in K$, according to definition (5.27), we have

$$C_{\rho_i/2}\cap F\cap B_\sigma(0) \;=\; \emptyset\,.$$

Since (5.20) holds and $\eta r=\rho_i/8$, the intersection of W_i+z_i with $C_{\rho_i/4}(x_i)\cap B_\sigma(0)$ must be empty. An easy computation yields that this is possible only if

$$(z_i+W_i)\cap C_{2\rho_i}(x_i)\cap B_{\sigma/2}(0)$$

contains a segment S_i of length $c_1\sigma$, where c_1 is a geometric constant; cf. Figure 6.

Therefore, there is a second geometric constant c_2 such that on the segment S_i we can find

$$N \;\ge\; c_2\sigma/\rho_i$$

points $\{z_i^j\}_{j=1,\dots,N}$ such that the balls $B_{\rho_i/2}(z_i^j)$ are pairwise disjoint. Recall that from (5.25) we have

$$\rho_i \;\le\; \sigma\eta$$

and thus we conclude

$$N \;\ge\; \frac{c_2}{\eta}\,. \tag{5.29}$$

By (5.20), each ball $B_{\rho_i/8}(z_i^j)$ must contain a point $w_i^j\in F$. Therefore, from the density lower bound (5.15) we have

$$\mathscr{H}^k\big(E\cap B_{\rho_i/8}(w_i^j)\big) \;\ge\; \frac{\delta\rho_i^k}{8^k}\,. \tag{5.30}$$

From our considerations, it follows that the balls $\{B_{\rho_i/8}(w_i^j)\}_{i=1,\dots N}$ are also pairwise disjoint and contained in $C_{4\rho_i}(x_i)$. Since the cylinders $\{C_{4\rho_i}(x_i)\}_{i\in K}$ are pairwise disjoint, we conclude that the family of balls

$$\big\{B_{\rho_i/8}(w_i^j) \;:\; i\in K\;,\; j=1,\dots,N\big\}$$

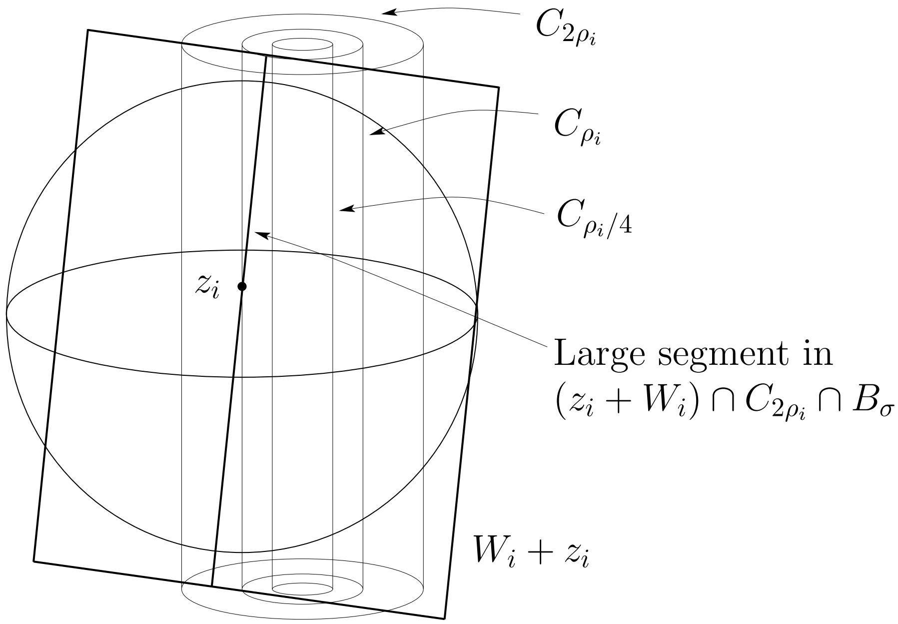

FIGURE 6. Since the intersection of $W_i + z_i$ with $C_{\rho_i/4}(x_i) \cap B_\sigma(0)$ is empty, $(z_i + W_i) \cap C_{2\rho_i}(x_i) \cap B_{\sigma/2}(0)$ contains a large segment of size comparable to σ.

are pairwise disjoint.

Therefore, from (5.29) and (5.30) we conclude

$$\begin{aligned} \mathscr{H}^k\big(E \cap B_\sigma(0)\big) \;&\geq\; \sum_{i\in K}\sum_{j=1}^N \mathscr{H}^k\big(E \cap B_{\rho_i/8}(z_i^j)\big) \;\geq\; \sum_{i\in K}\sum_{j=1}^N \frac{\delta\rho_i^k}{8^k} \\ &=\; \frac{N\delta}{8^k\omega_k}\sum_{i\in K}\omega_k r_i^k \;\geq\; \frac{c_2\delta}{8^k\omega_k\eta}\sum_{i\in K}\omega_k r_i^k\,. \end{aligned}$$

Recalling (5.28), this yields a positive constant c_3 (independent of δ and η) such that

$$\mathscr{H}^k\big(E \cap B_\sigma(0)\big) \;\geq\; c_3\frac{\delta}{\eta}r^k\,.$$

Therefore, we can choose η so small that we obtain a contradiction to (5.22). This completes the proof. $\square$

3. Proof of Theorem 5.1

PROOF. First of all, from Proposition 2.16, it follows that $\mu = f\mathscr{H}^k \llcorner E$ for some Borel set E and some nonnegative Borel function f. Our goal is to prove that $E \cap \{f > 0\}$ is rectifiable. In order to do this it suffices to show that this is the case for $E_c := E \cap \{c^{-1} \geq f \geq c\}$ for any $1 > c > 0$.

Fix $c \in]1, 0[$, set $F := E_c$ and define $\nu := \mathscr{H}^k \llcorner F$. Then, by the Besicovitch Differentiation Theorem and from Proposition 3.12 we have

$$\theta^{*k}(F, x) \;=\; \frac{\theta^{*k}(\mu, x)}{f(x)} \qquad \text{and} \qquad \theta^k_*(F, x) \;=\; \frac{\theta^k_*(\mu, x)}{f(x)}\,.$$

$$\mathrm{Tan}_k(\nu, x) \;\subset\; \mathrm{Tan}_k(\mu, x)/f(x)\,.$$

Therefore, for μ–a.e. x we have that

$$\infty \;>\; \theta^{*k}(F, x) \;\geq\; \theta^k_*(F, x) \;>\; 0 \tag{5.31}$$

$$\mathrm{Tan}_k\big(\nu, x\big) \;\subset\; \big\{a\mathscr{H}^k \llcorner V \;:\; a \geq 0 \quad \text{and} \quad V \text{ is a } k\text{–dimensional linear plane}\big\}\,. \tag{5.32}$$

We now prove that at every point x which satisfies (5.31) and (5.32), F is weakly linearly approximable.

Let us fix an x where (5.31) and (5.32) hold and assume by contradiction that F is not weakly linearly approximable at x.

Without loss of generality we can assume $x = 0$. Then there exists a positive η and a sequence $r_j \downarrow 0$ such that:

- either
$$\mathscr{H}^k\big(F \cap B_{r_j}(0) \setminus \big\{z : \mathrm{dist}\,(W, z) \leq \eta r_j\big\}\big) \;\geq\; \eta r_j^k \tag{5.33}$$
for every k–dimensional plane W and every j;

- or, for every k–dimensional plane W and for every j, there exists $z_{j,W} \in W$ with
$$\lim_{r_j \downarrow 0} \mathscr{H}^k \frac{\big(F \cap B_{\eta r_j}(z_{j,W})\big)}{r_j^k} \;=\; 0\,. \tag{5.34}$$

Set $\nu_j := r_j^{-k}\nu_{0,r_j}$. Since $\theta^{*k}(\nu, x) < \infty$, we can assume that a subsequence (not relabeled) of $\{\nu_j\}$ converges to $\nu_\infty \in \mathrm{Tan}_k(\nu, x)$. From (5.32) it follows that for some k–dimensional linear plane W and some constant $\bar{c} \geq 0$ we have $\nu_\infty = \bar{c}\mathscr{H}^k \llcorner W$.

From the definition of ν_j, (5.33) would translate into

$$\nu_j\Big(B_1(0) \setminus \big\{z : \mathrm{dist}\,(W, z) \leq \eta\big\}\Big) \;\geq\; \eta\,. \tag{5.35}$$

Consider the set Ω which is the closure of $B_1(0) \setminus \big\{z : \mathrm{dist}\,(W, z) \leq \eta\big\}$. From Proposition 2.7, we have

$$\bar{c}\mathscr{H}^k\big(W \cap \Omega\big) \;=\; \nu_\infty(\Omega) \;\geq\; \limsup_{j\uparrow\infty} \nu_j(\Omega) \;\geq\; \eta\,,$$

which is a contradiction because $W \cap \Omega = \emptyset$.

Similarly, (5.34) would translate into the existence of a sequence of points $x_j \in W \cap B_1(0)$ such that

$$\lim_{j\uparrow\infty} \nu_j(B_\eta(x_j)) \;=\; 0\,.$$

Passing to a subsequence we can assume that $x_j \to x \in W$. Therefore, we would conclude

$$\bar{c}\eta^k \;=\; c\mathscr{H}^k(W \cap B_\eta(x)) \;=\; \nu_\infty(B_\eta(x)) \;\leq\; \lim_{j\uparrow\infty} \nu_j(B_\eta(x_j)) \;=\; 0\,. \tag{5.36}$$

On the other hand, by Proposition 2.7 we have

$$\theta^k_*(F, x)\omega_k\rho^k \;\leq\; \nu_\infty(B_\rho(0)) \;=\; \bar{c}\rho^k \qquad \text{for } \mathscr{L}^1\text{–a.e. } \rho > 0.$$

From (5.31) we conclude that $\bar{c} > 0$, which contradicts (5.36). This concludes the proof. $\square$

CHAPTER 6

An overview of Preiss' proof

In this chapter and in the forthcoming three ones we will give a proof of Preiss' Theorem, which is outlined below.

THEOREM 6.1. *Let m be an integer and μ a locally finite measure on $\mathbb{R}^n$ such that $0 < \theta^m_*(\mu, x) = \theta^{m*}(\mu, x) < \infty$ for μ–a.e. x. Then μ is an m–dimensional rectifiable measure.*

Note that the cases $m = 0$ and $m = n$ are trivial. In the case $m = 1$ and $n = 2$, the Theorem was first proved by Besicovitch in his pioneering work [**2**]. More precisely, Besicovitch proved it for measures of the form $\mathscr{H}^1 \llcorner E$, when E is a Borel set $\mathscr{H}^1(E) < \infty$ and his proof was later extended to planar Borel measures by Morse and Randolph in [**24**]. In [**23**], Moore extended the result to the case $m = 1$ and arbitrary n. The general case was open for a long time until Preiss solved it completely in [**25**].

Marstrand Approach. Recalling Proposition 3.4, the assumption of Theorem 6.1 yields the following:

$$\mathrm{Tan}_m(\mu, x) \subset \big\{\theta(\mu, x)\nu \ : \ \nu \in \mathcal{U}^m(\mathbb{R}^n)\big\} \qquad \text{for } \mu\text{–a.e. } x. \tag{6.1}$$

Hence, if the following conjecture were true, we could apply Theorem 5.1 to conclude that μ is rectifiable.

CONJECTURE 6.2. *If $\nu \in \mathcal{U}^m(\mathbb{R}^n)$ then there exists an m–dimensional linear plane W such that $\nu = \mathscr{H}^m \llcorner W$.*

Such a conjecture is quite easy to prove when $m = 1$, and therefore, combined with Proposition 3.4, it yields a proof of Theorem 6.1 for $m = 1$. This proof differs from the "classical" proof of Besicovitch–Moore, which heavily relies on the fact that connected $\mathscr{H}^1$–finite sets are rectifiable. This new approach to the problem was introduced by Marstrand in [**16**], though not using the language of tangent measures. In that paper Marstrand proved the following theorem when $m = 2$ and $n = 3$:

THEOREM 6.3. *Let m be an integer and $E \subset \mathbb{R}^n$ be a Borel set such that $\mathscr{H}^m(E) < \infty$. If the density $\theta^m(E, x)$ exists and is equal to 1 at $\mathscr{H}^m$–a.e. $x \in E$, then E is rectifiable.*

In the language of tangent measures, the idea of the proof of Theorem 6.3 is that, for $\mathscr{H}^m$–a.e. x, the tangent measures to $\mathscr{H}^m \llcorner E$ enjoy a stronger property than just belonging to $\mathcal{U}^m(\mathbb{R}^n)$. This property allows to show that any such measure is of the form $\mathscr{H}^m \llcorner W$, and therefore makes it possible to apply the Marstrand–Mattila Rectifiability Criterion. This approach was completed in the general case by Mattila, see [**18**].

When $m = 2$ it is shown in [**25**] that the answer to Conjecture 6.2 is positive and therefore Marstrand's approach can be completed even for Theorem 6.1. However, we will see that the proof of this requires considerable work. When $m \geq 3$ Conjecture 6.2 turns out to be wrong, as is seen in the following example:

EXAMPLE 6.4. *Let Γ be the 3–dimensional cone of $\mathbb{R}^4$ given by*

$$\left\{x_4^2 \;=\; x_1^2 + x_2^2 + x_3^2\right\}.$$

Then $\mathscr{H}^3 \llcorner \Gamma \;\in\; \mathcal{U}^3(\mathbb{R}^4)$. We refer to Section 1 for the explicit calculations.

As we will see, there is a way to overcome this obstacle which finally leads to a proof of Theorem 6.1 in the general case.

Part A of Preiss' strategy. First, let us recall the following corollary of the argument that Marstrand used to prove Theorem 3.1 (cf. Corollary 3.9):

COROLLARY 6.5. *Let m be an integer and $\mu \in \mathcal{U}^m(\mathbb{R}^n)$. Then there exist an m–dimensional linear plane $V \subset \mathbb{R}^n$ and two sequences $\{x_i\}$ and $\{r_i\}$ such that*

$$\frac{\mu_{x_i,r_i}}{r_i^m} \overset{*}{\rightharpoonup} \mathscr{H}^m \llcorner V \qquad \textit{in the sense of measures.}$$

The first step towards the proof of Theorem 6.1 is then the following Lemma:

LEMMA 6.6. *Let μ be as in Theorem 6.1. Then for μ–a.e. x the following holds:*

(P) *If $\nu \in \mathrm{Tan}_m(\mu, x)$, then $r^{-m}\nu_{y,r} \in \mathrm{Tan}_m(\mu, x)$ for every $y \in \mathrm{supp}\,(\nu)$ and $r > 0$.*

REMARK 6.7. *From the definition of tangent measure it follows easily that*

$$\nu \in \mathrm{Tan}_m(\mu, x) \implies r^{-m}\nu_{0,r} \in \mathrm{Tan}_m(\mu, x) \qquad \textit{at } \text{every } x \textit{ and for every } r > 0.$$

Note, however, that (P) is much stronger and it cannot be expected to hold at every point x. For instance, if we take the cone C of Example 6.4 and we set $\mu := \mathscr{H}^3 \llcorner C$, then $\mathrm{Tan}_3(\mu, 0) = \{\mu\}$, whereas it is clear that for any $x \neq 0$ and every $r > 0$ we have $r^{-3}\mu_{x,r} \neq \mu$.

Proposition 3.4, Corollary 6.5, and Lemma 6.6 yield the following Theorem, which was first proved by Marstrand in [**17**].

THEOREM 6.8 (Part A). *Let μ be as in Theorem 6.1, then for μ–a.e. $x \in \mathbb{R}^n$ there exists a plane W_x such that $\theta(\mu, x)\mathscr{H}^m \llcorner W_x \in \mathrm{Tan}_m(\mu, x)$.*

In other words, in spite of the existence of Example 6.4, we conclude that at almost every point x, the set of tangent measures contains *at least* one plane.

Part B of Preiss' strategy. Let us first introduce some notation which will be very useful in the rest of these notes.

DEFINITION 6.9. *We denote by $G(m, n)$ the set of m–dimensional planes V of $\mathbb{R}^n$ and by $\mathcal{G}_m(\mathbb{R}^n)$ the set*

$$\mathcal{G}_m(\mathbb{R}^n) \;:=\; \left\{\mathscr{H}^m \llcorner V \;:\; V \in G(m, n)\right\}.$$

We call the measures of $\mathcal{G}_m(\mathbb{R}^n)$ flat *measures.*

Taking into account Theorem 6.8, Theorem 6.1 will then follow from Theorem 6.10 below.

THEOREM 6.10 (Part B). *Let μ be as in Theorem 6.1 and x a point such that*

- $\mathrm{Tan}_m(\mu, x) \subset \theta(\mu, x)\mathcal{U}^m(\mathbb{R}^n)$;
- $\mathrm{Tan}_m(\mu, x)$ *contains a measure of the form* $\theta(\mu, x)\mathcal{H}^m \llcorner V$ *for some* m*–dimensional plane* V.

Then $\mathrm{Tan}_m(\mu, x) \subset \theta(\mu, x)\mathcal{G}_m(\mathbb{R}^n)$.

In other words, Theorem 6.10 says that, if the set of tangent measures to μ at the point x contains a plane, then *any* tangent measure to μ at x must be a plane. Theorem 6.8 and Theorem 6.10 imply that at μ–a.e. x, the set of tangent measures consists of k–dimensional planes. Therefore, we can apply Theorem 5.1 to conclude that μ is rectifiable.

A very sketchy outline of the strategy of Preiss' proof of Theorem 6.10 is the following:

- The set of m–uniform measures can be divided into two subsets, given by $\mathcal{G}_m(\mathbb{R}^n)$ and its complementary $\mathcal{U}^m(\mathbb{R}^n) \setminus \mathcal{G}_m(\mathbb{R}^n)$.

- If $\mu \in \mathcal{U}^m(\mathbb{R}^n) \setminus \mathcal{G}_m(\mathbb{R}^n)$, then on very large balls μ must be quite different from a flat measure (i.e. it must be "curved at infinity"). This translates into the fact that $\mathcal{G}_m(\mathbb{R}^n)$ is, in some sense, disconnected from $\mathcal{U}^m(\mathbb{R}^n) \setminus \mathcal{G}_m(\mathbb{R}^n)$.

- On the other hand $\mathrm{Tan}_m(\mu, x)$ enjoys some "connectedness" properties, just from the way it is defined: It is the set of blow–ups of the *same measure* at the *same point*. Therefore, it cannot happen that $\mathrm{Tan}_m(\mu, x)$ contains at the same time an element of $\theta(\mu, x)\mathcal{G}_m(\mathbb{R}^n)$ and one of $\theta(\mu, x)[\mathcal{U}^m(\mathbb{R}^n) \setminus \mathcal{G}_m(\mathbb{R}^n)]$.

In order to exploit these ideas we will need this and the next three chapters. More precisely, Theorem 6.10 will be split into the three key Propositions 6.16, 6.18, and 6.19. The proof of each proposition is contained in one of the next three chapters, whereas in this chapter we will show how they imply Theorem 6.10.

Plan of the chapter. In Section 1 we prove that the measure of Example 6.4 is flat. In section 2 we prove Lemma 6.6 and Theorem 6.8. In Section 3 we introduce some definitions and we state the three Propositions 6.16, 6.18, and 6.19 which are the three main steps for proving Theorem 6.10. In Section 4 we show how Theorem 6.10 follows from these Propositions.

1. The cone $\{x_4^2 = x_1^2 + x_2^2 + x_3^2\}$

In this section we will set $\Gamma := \{x \in \mathbb{R}^4 : x_4^2 = x_1^2 + x_2^2 + x_3^2\}$. Our goal is to prove the following

PROPOSITION 6.11. *The measure* $\mathcal{H}^3 \llcorner \Gamma$ *belongs to* $\mathcal{U}^3(\mathbb{R}^4)$.

A direct proof of this proposition can be found in the paper [**11**]. Indeed in this paper the authors, using differential geometric arguments, show the following complete classification result.

THEOREM 6.12. $\mu \in \mathcal{U}^{m-1}(\mathbb{R}^m)$ *if and only if* μ *is flat or* $m \geq 4$ *and there exists an orthonormal system of coordinates such that* $\mu = \mathcal{H}^{m-1} \llcorner \{x_1^2 + x_2^2 + x_3^2 = x_4^2\}$.

Here we propose a proof of Proposition 6.11 which is less direct. This proof is not much longer than that given in [**11**] and it exploits some calculations and tricks that will be used again in the proof of Theorem 6.10. However, we first need the following definition and the subsequent technical lemma:

DEFINITION 6.13. *Let $\mathcal{P}$ be the set of polynomials of one real variable. Then we let $\mathcal{R}_n$ be the vector space generated by*

$$\left\{ f \in C^\infty(\mathbb{R}^n) \, : \, f(x) = a + P(|x|^2)e^{-b|x|^2} \quad \text{where } P \in \mathcal{P},\, a \in \mathbb{R},\, b > 0 \right\}.$$

The proof of the following lemma is a straightforward application of the Stone–Weiestrass Theorem. We include its proof for the reader's convenience.

LEMMA 6.14. *Let $g \in C_c(\mathbb{R})$ and define $G \in C_c(\mathbb{R}^n)$ as*

$$G(x) \; := \; g(|x|)\,.$$

Then there exists a sequence $\{G_k\} \subset \mathcal{R}_n$ such that $G_k \to G$ uniformly on the whole $\mathbb{R}^n$.

PROOF. Clearly $G \in \mathcal{R}_n$ if and only if $G(x) = g(|x|)$ for some $g \in \mathcal{R}_1$. Therefore, it suffices to prove the lemma when $n = 1$.

Let $[0, \infty]$ be the one–point compactification of $[0, \infty[$ and note that every $f \in \mathcal{R}_1$ extends to a unique function $\tilde{f} \in C([0, \infty])$. We denote by $\tilde{\mathcal{R}}$ the vector space given by the continuous extensions of functions of $\mathcal{R}_1$.

Since $\mathcal{R}_1$ is an algebra of functions, the same holds for $\tilde{\mathcal{R}}$. Moreover, note that

- For every $a, b \in [0, \infty]$ there exists a function $f \in \tilde{\mathcal{R}}$ such that $f(a) \neq f(b)$.
- For every $a \in [0, \infty]$ there exists a function $f \in \tilde{\mathcal{R}}$ such that $f(a) \neq 0$.

Therefore, we can apply the Stone–Weierstrass Theorem to show that $\tilde{\mathcal{R}}$ is dense in $C([0, \infty])$. This concludes the proof. $\square$

PROOF OF PROPOSITION 6.11. **Step 1** In the next steps we will prove that (claim) for every $p \in \Gamma \setminus \{0\}$ there exists a constant c_p such that

$$\mathscr{H}^3\big(B_r(p) \cap \Gamma\big) \; = \; c_p r^3 \qquad \forall r > 0\,. \tag{6.2}$$

This claim suffices to prove the Proposition. Indeed fix $p \in \Gamma \setminus \{0\}$ and recall that Γ is a C^1 manifold in a neighborhood of p. Therefore,

$$c_p \; = \; \lim_{r \downarrow 0} \frac{\mathscr{H}^3\big(B_r(p) \cap \Gamma\big)}{r^3} \; = \; \omega_3\,,$$

and hence $\mathscr{H}^3\big(B_r(p) \cap \Gamma\big) = \omega_3 r^3$ for every $r > 0$ and $p \in \Gamma \setminus \{0\}$. On the other hand, if we fix $r > 0$, we can take a sequence $\{p_i\} \subset \Gamma \setminus \{0\}$ such that $p_i \to 0$ in order to conclude that

$$\mathscr{H}^3\big(\Gamma \cap B_r(0)\big) \; = \; \lim_{p_i \to 0} \mathscr{H}^3\big(\Gamma \cap B_r(p_i)\big) \; = \; \omega_3 r^3\,.$$

Step 2 Let us fix $p \in \Gamma \setminus \{0\}$. In order to prove (6.2) it suffices to show that for every function $\varphi \in C_c(\mathbb{R})$, there exists a constant c_φ such that

$$\int \varphi\left(\frac{|x-p|}{r}\right) d\mathscr{H}^3 \llcorner \Gamma(x) \; = \; c_\varphi r^3 \qquad \text{for every } r > 0. \tag{6.3}$$

Indeed, knowing (6.3) we could take a sequence of $\varphi_i \in C_c(B_1(0))$ which converges pointwise everywhere to $\mathbf{1}_{B_1(0)}$. In this case, the constants c_{φ_i} would be uniformly bounded and passing to a subsequence, not relabeled, we could assume that they converge to some constant c_p. Plugging φ_i in (6.3) and passing to the limit in i we would conclude that $\mathscr{H}^3(\Gamma \cap B_r(0)) = c_p r^3$.

Now, let $\mathcal{B}$ be the set of functions $f \in L^1(\mathbb{R}^4, \mathscr{H}^3 \llcorner \Gamma)$ such that

- $f(x) = \varphi(|x-p|)$;
- There exists a constant c_φ which satifsies (6.3).

Clearly this set is a vector space. We claim that it would suffice to show

$$e^{-|x-p|^2} \;\in\; \mathcal{B}\,, \tag{6.4}$$

in order to conclude (6.3).

Indeed, assume for the moment that (6.4) holds and take the derivative in r of the equality

$$\int e^{-|x-p|^2/r^2}\, d\mathscr{H}^3 \llcorner \Gamma(x) \;=\; cr^3$$

to obtain

$$\int \frac{2|x-p|^2}{r^3} e^{-|x-p|^2/r^2}\, d\mathscr{H}^3 \llcorner \Gamma(x) \;=\; cr^2\,.$$

From this we conclude that $|x-p|^2 e^{-|x-p|^2}$ belongs to $\mathcal{B}$. Taking a second derivative we find

$$(|x-p|^4 + |x-p|^2) e^{-|x-p|^2} \;\in\; \mathcal{B}\,,$$

and hence we conclude $|x-p|^4 e^{-|x-p|^2} \in \mathcal{B}$. By induction, we obtain $(|x-p|^2)^k e^{-|x-p|^2} \in \mathcal{B}$ for every positive integer k. Therefore, (6.4) would imply that, if P is a polynomial, then $P(|x-p|^2)e^{-|x-p|^2} \in \mathcal{B}$. A change of variables implies that every function of the following type belongs to $\mathcal{B}$:

$$P(|x-p|^2)e^{-a|x-p|^2} \qquad \text{where } P \text{ is a polynomial and } a > 0.$$

By linearity, we conclude that for every $\gamma \in \mathcal{R}_n$, the function $g(x) := \gamma(x-p)$ belongs to $\mathcal{B}$.

Now fix $f \in C_c(\mathbb{R}^4)$ of the form $\varphi(|x-p|)$. Clearly, $e^{|x-p|^2}\varphi(|x-p|)$ is still a continuous compactly supported function. Then using Lemma 6.14 we conclude that there exists a sequence of functions $\{\gamma_k\} \subset \mathcal{R}_n$ such that the functions $f_k(x) = \gamma_k(x-p)$ converge uniformly to $e^{|x-p|^2}\varphi(|x-p|)$. Therefore, for every fixed $r > 0$ we could compute

$$\int \varphi\left(\frac{|x-p|}{r}\right) d\mathscr{H}^3 \llcorner \Gamma(x) \;=\; \lim_{k\uparrow\infty} \int e^{-|x-p|^2/r^2} \gamma_k(x-p)\, d\mathscr{H}^3 \llcorner \Gamma(x)\,.$$

On the other hand $e^{-|x|^2/r^2}\gamma_k(x) \in \mathcal{R}_n$ and hence $e^{-|x-p|^2/r^2}\gamma_k(x-p) \in \mathcal{B}$. This means that

$$\int \varphi\left(\frac{|x-p|}{r}\right) d\mathscr{H}^3 \llcorner \Gamma \;=\; c_\varphi r^3$$

for some constant c_φ.

Step 3 It remains to prove (6.4), that is

$$I(r) \;:=\; \int e^{-r^2|x-p|^2}\, d\mathscr{H}^3 \llcorner \Gamma(x) \;=\; cr^{-3} \qquad \text{for every } r > 0. \tag{6.5}$$

Note that the cone Γ is invariant for dilations centered at the origin and rotations that keep $(0,0,0,1)$ fixed. Therefore, it suffices to show (6.5) when $p=(1,0,0,1)$. We compute

$$I(r) \;=\; \int_0^\infty \int_{\Gamma\cap\partial B_\rho(0)} e^{-r^2\left[(x_1-1)^2+x_2^2+x_3^2+(x_4-1)^2\right]}\, d\mathscr{H}^2(x)\, d\rho \;=:\; \int_0^\infty J(\rho)\, d\rho\ .$$

Note that $(x_1-1)^2+x_2^2+x_3^2+(x_4-1)^2 \;=\; |x|^2+2-2(x_1+x_4)$ and that $\Gamma\cap\partial B_\rho(0)$ is given by

$$\left\{x_4=\rho/\sqrt{2}\,,\ x_1^2+x_2^2+x_3^2=\rho^2/2\right\}\cup\left\{x_4=-\rho/\sqrt{2}\,,\ x_1^2+x_2^2+x_3^2=\rho^2/2\right\}.$$

Therefore, we can compute

$$\begin{aligned} J(\rho) \;&=\; e^{-r^2(\rho^2+2)}\left(e^{\sqrt{2}r^2\rho}+e^{-\sqrt{2}r^2\rho}\right)\int_{x_1^2+x_2^2+x_3^2=\rho^2/2} e^{2r^2x_1}\, d\mathscr{H}^2\\ &=:\; e^{-r^2(\rho^2+2)}\left(e^{\sqrt{2}r^2\rho}+e^{-\sqrt{2}r^2\rho}\right)K(\rho)\,. \end{aligned}$$

We use the spherical coordinates $(\theta,\phi)\to(\cos\theta,\sin\theta\sin\phi,\sin\theta\cos\phi)$ to compute

$$\begin{aligned} K(\rho) \;&=\; \frac{\rho^2}{2}\int_0^\pi e^{-\sqrt{2}r^2\rho\cos\theta}\,2\pi\sin\theta\, d\theta \;=\; \frac{\pi\rho}{\sqrt{2}r^2}\int_0^\pi e^{-\sqrt{2}r^2\rho\cos\theta}\left(\sqrt{2}r^2\rho\sin\theta\right) d\theta\\ &=\; \frac{\pi\rho}{\sqrt{2}r^2}\, e^{-\sqrt{2}r^2\rho\cos\theta}\Big|_0^\pi \;=\; \frac{\pi\rho}{\sqrt{2}r^2}\left[e^{\sqrt{2}r^2\rho}-e^{-\sqrt{2}r^2\rho}\right]. \end{aligned}$$

Hence, we conclude

$$\begin{aligned} J(\rho) \;&=\; \frac{\pi\rho}{\sqrt{2}r^2}e^{-r^2(\rho^2+2)}\left(e^{\sqrt{2}r^2\rho}+e^{-\sqrt{2}r^2\rho}\right)\left(e^{\sqrt{2}r^2\rho}-e^{-\sqrt{2}r^2\rho}\right)\\ &=\; \frac{\pi\rho}{\sqrt{2}r^2}e^{-r^2(\rho^2+2)}\left(e^{2\sqrt{2}r^2\rho}-e^{-2\sqrt{2}r^2\rho}\right) \;=\; \frac{\pi\rho}{\sqrt{2}r^2}\left(e^{-r^2(\rho-\sqrt{2})^2}-e^{-r^2(\rho+\sqrt{2})^2}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} I(r) \;&=\; \frac{\pi}{\sqrt{2}r^2}\left[\int_0^\infty e^{-r^2(\rho-\sqrt{2})^2}\rho\, d\rho-\int_0^\infty e^{-r^2(\rho+\sqrt{2})^2}\rho\, d\rho\right]\\ &=\; \frac{\pi}{\sqrt{2}r^2}\left[\int_{-\sqrt{2}}^\infty e^{-r^2t^2}(t+\sqrt{2})\, dt-\int_{\sqrt{2}}^\infty e^{-r^2t^2}(t-\sqrt{2})\, dt\right]\\ &=\; \frac{\pi}{\sqrt{2}r^2}\left\{\int_{-\sqrt{2}}^{\sqrt{2}} e^{-r^2t^2}t\, dt+\sqrt{2}\left[\int_{-\sqrt{2}}^\infty e^{-r^2t^2}\, dt+\int_{\sqrt{2}}^\infty e^{-r^2t^2}\, dt\right]\right\}\\ &=\; \frac{\pi}{r^2}\left[\int_{-\sqrt{2}}^\infty e^{-r^2t^2}\, dt+\int_{-\infty}^{-\sqrt{2}} e^{-r^2t^2}\, dt\right] \;=\; \frac{\pi^{3/2}}{r^3}\,. \end{aligned}$$

This concludes the proof. $\square$

2. Part A of Preiss' strategy

The aim of this section is to prove Lemma 6.6. In order to do this, it is very convenient to use the metric d on the space of measures $\mathcal{M}$ introduced in Proposition 2.6, which induces the topology of the weak* convergence on bounded subsets of $\mathcal{M}$.

Proof of Lemma 6.6. It is enough to prove that the following condition holds for μ–a.e. a:

(R) If $\nu \in \mathrm{Tan}_m(\mu, a)$ and $x \in \mathrm{supp}\,(\nu)$ then $\nu_{x,1} \in \mathrm{Tan}_m(\mu, a)$.

Indeed, let a be a point where (R) holds and fix $b \in \mathbb{R}^n$, $\zeta \in \mathrm{Tan}(\mu, a)$, and $r > 0$. Note that $\nu := r^{-m}\zeta_{0,r} \in \mathrm{Tan}_m(\mu, a)$ and that $r^{-m}\zeta_{b,r} = \nu_{b/r,1}$. Hence, applying (R) to $x = b/r \in \mathrm{supp}\,(\nu)$, we conclude that $r^{-m}\zeta_{b,r} \in \mathrm{Tan}_m(\mu, x)$.

In order to prove (R), for every $k, j \in \mathbb{N}$ we define

$$A_{k,j} \;:=\; \Big\{a \in \mathbb{R}^n \;:\; \exists \nu^a \in \mathrm{Tan}_m(\mu, a) \quad \text{and} \quad x_a \in \mathrm{supp}\,(\nu^a) \quad \text{such that}$$
$$d\big(r^{-m}\mu_{a,r}, \nu^a_{x_a,1}\big) \;\geq\; \frac{1}{k} \quad \forall r < \frac{1}{j}\Big\}.$$

Clearly it suffices to show that $\mu(A_{k,j}) = 0$. We argue by contradiction and assume that $\mu(A_{k,j}) > 0$ for some k and j. For some $R > 0$ we have that the set

$$B_{k,j} \;:=\; A_{k,j} \cap \big\{a \;:\; R^{-1} < \theta^m_*(\mu, a) = \theta^{*m}(\mu, a) \leq R\big\}$$

has positive measure. We drop the indices from $B_{k,j}$ and we consider the set

$$\mathcal{S} \;:=\; \big\{\nu^a_{x_a,1}\big\}_{a \in B}\,.$$

Note that $\mathcal{S}$ is a set of uniformly locally bounded measures. Indeed recall that $\nu^a_{x_a,1}(B_r(0)) = \nu^a(B_r(0)) = \theta(\mu, a)r^m$.

We cover $\mathcal{S}$ with a countable family of sets G_i of type

$$G_i \;:=\; \left\{\zeta \;:\; d(\zeta, \zeta_i) < \frac{1}{4k}\right\}.$$

Clearly for some i we have

$$\mu\Big(\big\{a \in B \;:\; \nu^a_{x_a,1} \in G_i\big\}\Big) \;>\; 0\,.$$

Therefore, if set $A := G_i$, we have that

- $\mu(A) > 0$;
- For every $a \in A$ there exists $\nu^a \in \mathrm{Tan}_m(a, \mu)$ with

$$d\big(r^{-m}\mu_{a,r}, \nu^a_{x_a,1}\big) \;\geq\; \frac{1}{k} \qquad \text{for every } r < \frac{1}{j} \tag{6.6}$$

  and

$$d\big(\nu^a_{x_a,1}, \nu^b_{x_b,1}\big) \;<\; \frac{1}{2k} \qquad \text{for every } a, b \in A. \tag{6.7}$$

We choose:

- $a \in A$ such that

$$\lim_{r \downarrow 0} \frac{\mu(A \cap B_r(a))}{\mu(B_r(a))} \;=\; 1\,; \tag{6.8}$$

- $r_i \downarrow 0$ such that

$$r_i^{-m}\mu_{a,r_i} \;\stackrel{*}{\rightharpoonup}\; \nu^a\,; \tag{6.9}$$

- $a_i \in A$ such that

$$\left|a_i - (a + r_i x_a)\right| < \operatorname{dist}(a + r_i x_a, A) + \frac{r_i}{i}. \tag{6.10}$$

Note that

$$\lim_{i\uparrow\infty} \frac{\operatorname{dist}(a + r_i x_a, A)}{r_i} = 0. \tag{6.11}$$

Indeed, if we had

$$\limsup_{i\uparrow\infty} \frac{\operatorname{dist}(a + r_i x_a, A)}{r_i} > c,$$

then from (6.8) we would obtain $\nu^a(B_c(0)) = 0$, which is clearly a contradiction to $\theta_*^m(\mu, a) > 0$.

Note that

$$r_i^{-m}\mu_{a_i,r} = \left(r_i^{-m}\mu_{a,r_i}\right)_{\frac{a_i-a}{r_i},1} \stackrel{*}{\rightharpoonup} \nu^a_{x_a,1}.$$

Therefore, for $r_i < j^{-1}$ sufficiently small we have

$$d\left(\nu^a_{x_a,1}, r_i^{-m}\mu_{a_i,r_i}\right) < \frac{1}{2k}. \tag{6.12}$$

On the other hand, since $a_i \in A$ we have

$$\frac{1}{k} < d\left(\nu^{a_i}_{x_{a_i},1}, r_i^{-m}\mu_{a_i,r_i}\right). \tag{6.13}$$

From the triangle inequality we obtain

$$d\left(\nu^{a_i}_{x_{a_i},1}, r_i^{-m}\mu_{a_i,r_i}\right) \leq d\left(\nu^{a_i}_{x_{a_i},1}, \nu^a_{x_a,1}\right) + d\left(\nu^a_{x_a,1}, r_i^{-m}\mu_{a_i,r_i}\right). \tag{6.14}$$

From (6.12) we find that the second summand in the right hand side of (6.14) is strictly less than $(2k)^{-1}$. The same inequality holds for the first summand in view of (6.7). Therefore, we conclude

$$d\left(\nu^{a_i}_{x_{a_i},1}, r_i^{-m}\mu_{a_i,r_i}\right) < \frac{1}{k}$$

which contradicts (6.13). ☐

Proof of Theorem 6.8. Note that at a point x where Corollary 6.5 and Lemma 6.6 hold, we conclude that the weak* closure of $\operatorname{Tan}_m(\mu, x)$ contains a measure of type $\nu = \theta(\mu, x)\mathscr{H}^m \llcorner V$ where V is an m–dimensional linear plane. Now, from the definition it follows that $\operatorname{Tan}_m(\mu, x)$ is a weak* closed set. This completes the proof. ☐

3. Part B of Preiss' strategy: Three main steps

We begin by stating a definition of the tangent measures at infinity which is obtained by a scaling procedure which is the opposite of a blow up, namely a "blow down" of the original measure.

Definition 6.15. *Let $\alpha \in \mathbb{R}^+$ and μ be a locally finite measure. Then we define $\operatorname{Tan}_\alpha(\mu, \infty)$ as the set of measures ν such that there exists $r_i \uparrow \infty$ with*

$$\frac{\mu_{0,r_i}}{r_i^\alpha} \stackrel{*}{\rightharpoonup} \nu.$$

Note that when $\mu \in \mathcal{U}^m(\mathbb{R}^n)$, the family of measures

$$\left\{\frac{\mu_r}{r^m}\right\}_{r>0}$$

is locally uniformly bounded. Therefore, for every sequence $\{r_i\} \uparrow \infty$ we can extract a subsequence $r_{i(j)}$ such that $r_{i(j)}^{-m}\mu_{0,r_{i(j)}} \stackrel{*}{\rightharpoonup} \nu$ for some measure ν.

In Chapter 7 we will show that m–uniform measures have a unique tangent measure at infinity. Proposition 6.16 below provides the precise statement.

Proposition 6.16. *If $\nu \in \mathcal{U}^m(\mathbb{R}^n)$, then there exists $\zeta \in \mathcal{U}^m(\mathbb{R}^n)$ such that $\mathrm{Tan}_m(\nu,\infty) = \{\zeta\}$.*

This proposition means that the whole family of measures $\{r^{-m}\nu_{0,r}\}_{r>0}$ converges to ζ as $r \uparrow \infty$. Therefore, we will speak of *the* tangent measure at infinity to ν. Such a uniqueness property yields that the measure ζ is, in some sense, a "cone" and therefore it will enable us to draw many useful conclusions about its structure.

Definition 6.17. *We say that a measure $\nu \in \mathcal{U}^m(\mathbb{R}^n)$ is* flat at infinity *if the tangent measure at infinity is flat.*

In Chapter 8 we will show that, if $\nu \in \mathcal{U}^m(\mathbb{R}^n)$ and its tangent measure at infinity is sufficiently close to a flat measure, then ν is flat at infinity. More precisely we will prove the following:

Proposition 6.18. *There exists a constant $\varepsilon > 0$ which depends only on m and n such that:*

- *If $\nu \in \mathcal{U}^m(\mathbb{R}^n)$, $\{\zeta\} = \mathrm{Tan}_m(\nu,\infty)$, and*

$$\min_{V\in G(m,n)} \int_{B_1(0)} \mathrm{dist}^2\,(x,V)\,d\zeta(x) \le \varepsilon\,,$$

then ζ is flat.

Finally in the last chapter we will prove the following proposition.

Proposition 6.19. *If $\nu \in \mathcal{U}^m(\mathbb{R}^n)$ is flat at infinity, then ν is flat.*

4. From the three main steps to the proof of Theorem 6.10

In this section we will show how Theorem 6.10 follows from the three Propositions of the previous section. In order to do this we will need the following lemma.

Lemma 6.20. *Let $\varphi \in C_c(\mathbb{R}^n)$ and consider the functional $F : \mathcal{M}(\mathbb{R}^n) \to \mathbb{R}$ given by*

$$F(\mu) \;:=\; \min_{V\in G(m,n)} \int \varphi(z)\,\mathrm{dist}^2\,(z,V)\,d\mu(z)\,.$$

Then $F(\mu_i) \to F(\mu)$ if $\mu_i \stackrel{*}{\rightharpoonup} \mu$.

Fix $\varphi \in C_c(B_2(0))$ such that $0 \le \varphi \le 1$ and $\varphi = 1$ on $B_1(0)$. From the very definition of the functional F, a measure $\mu \in \mathcal{U}^m(\mathbb{R}^n)$ is flat if and only if $F(\mu) = 0$. The idea of the proof of Theorem 6.10 is then the following. Assume by contradiction that there exists a point x where a tangent measure α is flat and another tangent measure ν is not flat. Let χ

be the measure tangent at infinity to ν. Then from Proposition 6.19 we conclude that χ is not flat and from Proposition 6.18 we obtain that $F(\chi) > \varepsilon$. On the other hand $F(\alpha) = 0$. Define

$$f(r) \;:=\; F\big(r^{-m}\mu_{0,r}\big)\,.$$

Note that there exist $r_k \downarrow 0$ and $s_k \downarrow 0$ such that $r_k^{-m}\mu_{x,r_k} \overset{*}{\rightharpoonup} \alpha$ and $s_k^{-m}\mu_{x,s_k} \overset{*}{\rightharpoonup} \chi$. Hence, from Lemma 6.20 we conclude that $f(r_k) \downarrow 0$ and $\limsup_k f(s_k) > \varepsilon$. On the other hand, Lemma 6.20 implies that f is continuous. Therefore, the function f should have the oscillatory behavior sketched in Figure 1.

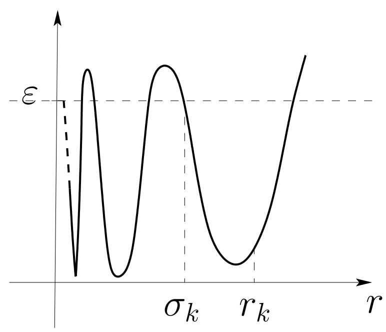

FIGURE 1. The graph of f

Clearly $f(r_k)$ will be below ε, for k large enough. Denote by σ_k the first point where f reaches again the level ε. One can show that $r_k/\sigma_k \uparrow \infty$. If we assume that $\sigma_k^{-m}\mu_{x,\sigma_k} \overset{*}{\rightharpoonup} \xi$ for some measure ξ, the condition $r_k/\sigma_k \uparrow \infty$ implies that there exists a sequence of points $\theta_k \in [\sigma_k, r_k]$ such that $\theta_k^{-m}\mu_{x,\theta_k}$ converges to the tangent measure to ξ at infinity. Since $f(\sigma_k) = \varepsilon$, ξ cannot be flat. On the other hand, since $f(\theta_k) \le \varepsilon$, Proposition 6.18 implies that the tangent measure to ξ at infinity is flat. Therefore, we find a contradiction to Proposition 6.19.

We will give the details of this argument after proving Lemma 6.20.

PROOF OF LEMMA 6.20. First of all, let V_i be such that

$$F(\mu_i) \;:=\; \int \varphi(z)\,\mathrm{dist}^2\,(z, V_i)\,d\mu_i\,.$$

Up to subsequences we can assume that V_i converges to an m–dimensional plane V_∞. Therefore, the functions $\varphi(\cdot)\,\mathrm{dist}^2\,(\cdot, V_i)$ converge uniformly to $\varphi(\cdot)\,\mathrm{dist}^2\,(\cdot, V_\infty)$ and we find that

$$\lim_{i\uparrow\infty} \int \varphi(z)\,\mathrm{dist}^2\,(z, V_i)\,d\mu_i \;=\; \int \varphi(z)\,\mathrm{dist}^2\,(z, V_\infty)\,d\mu\,.$$

This implies that

$$\liminf_{i\uparrow\infty} F(\mu_i) \;\ge\; F(\mu)\,.$$

Finally, let V be an m–dimensional plane such that

$$F(\mu) \;=\; \int \varphi(z)\,\mathrm{dist}^2\,(z, V)\,d\mu\,.$$

Since

$$\lim_{i\uparrow\infty}\int \varphi(z)\,\mathrm{dist}^2\,(z,V)\,d\mu_i \;=\; \int \varphi(z)\,\mathrm{dist}^2\,(z,V)\,d\mu\,,$$

we conclude that

$$\limsup_{i\uparrow\infty} F(\mu_i) \;\leq\; F(\mu)\,.$$

This concludes the proof. □

Proof of Theorem 6.10. We argue by contradiction and we fix a point x such that

- $\mathrm{Tan}(\mu,x)\subset\theta(\mu,x)\mathcal{U}^m(\mathbb{R}^n)$;
- There exists $\nu\in\mathrm{Tan}(\mu,x)$ such that $\nu/\theta(\mu,x)$ is flat;
- There exists $\zeta\in\mathrm{Tan}(\mu,x)$ such that $\nu/\theta(\mu,x)$ is not flat.

Without loss of generality we can assume that $\theta(\mu,x)=1$.

Now, let χ be the tangent measure to ν at infinity and fix a nonnegative $\varphi\in C_c(B_2(0))$ such that $\varphi=1$ on $B_1(0)$. Proposition 6.18 and Proposition 6.19 give that

$$F(\chi) \;>\; \varepsilon\,. \tag{6.15}$$

Note that $\chi\in\mathrm{Tan}(\mu,x)$. Now we fix $r_k\downarrow 0$ and $s_k\downarrow 0$ such that

$$\frac{\mu_{x,r_k}}{r_k^m}\;\stackrel{*}{\rightharpoonup}\;\nu \qquad \frac{\mu_{x,s_k}}{s_k^m}\;\stackrel{*}{\rightharpoonup}\;\chi\,.$$

We can also assume that $s_k<r_k$. Define

$$f(r) \;:=\; F\big(r^{-m}\mu_{0,r}\big)\,.$$

Since ν is flat, from Lemma 6.20 we have

$$\lim_{r_k\downarrow 0} f(r_k) \;=\; F(\nu) \;=\; 0\,.$$

Hence, for r_k sufficiently small, we have

$$f(r_k) \;<\; \varepsilon\,. \tag{6.16}$$

On the other hand, since

$$\lim_{s_k\downarrow 0} f(s_k) \;=\; F(\chi) \;>\; \varepsilon\,,$$

for s_k sufficiently small we have

$$f(s_k) \;>\; \varepsilon\,. \tag{6.17}$$

From Lemma 6.20 we conclude that f is a continuous function of r. Hence, we can fix $\sigma_k\in[s_k,r_k]$ such that $f(\sigma_k)=\varepsilon$ and $f(r)\leq\varepsilon$ for $r\in[\sigma_k,r_k]$. By compactness there exists a subsequence of $\{\sigma_k\}$, not relabeled, such that $\sigma_k^{-m}\mu_{x,\sigma_k}$ converges weakly* to a measure $\xi\in\mathcal{U}^m(\mathbb{R}^n)$. Clearly,

$$F(\xi) \;=\; \lim_{\sigma_k\downarrow 0} f(\sigma_k) \;=\; \varepsilon\,.$$

Hence, ξ cannot be flat. Now, note that $r_k/\sigma_k\to\infty$. Indeed, if for some subsequence, not relabeled, we had that r_k/σ_k converges to a constant C (necessarily larger than 1), we would conclude that

$$\frac{\xi_{0,C}}{C^m} \;=\; \nu$$

and hence ξ would be flat.

Next, note that for any given $R > 0$ we have

$$(R\sigma_k)^{-m}\mu_{x,R\sigma_k} \overset{*}{\rightharpoonup} R^{-m}\xi_{0,R}\,.$$

Hence,

$$F\big(\xi_{0,R}\big) \;=\; \lim_{k\uparrow\infty} f(R\sigma_k)\,.$$

If $R \geq 1$ we have $R\sigma_k \geq \sigma_k$. Moreover, since $r_k/\sigma_k \to \infty$, we conclude that $R\sigma_k \in [\sigma_k, r_k]$ whenever k is large enough. Therefore, we conclude that

$$F\big(R^{-m}\xi_{0,R}\big) \;\leq\; \varepsilon \qquad \text{for every } R \geq 1.$$

Let ψ be the tangent measure at infinity to ξ. Then

$$F(\psi) \;=\; \lim_{R\uparrow\infty} F\big(R^{-m}\xi_{0,R}\big) \;\leq\; \varepsilon\,.$$

Applying Proposition 6.18 we conclude that ψ is flat, and hence from Proposition 6.19 we conclude that ξ is flat, which is a contradiction. $\square$

CHAPTER 7

Moments and uniqueness of the tangent measure at infinity

In this chapter we will prove Proposition 6.16, that is the uniqueness of tangent measures at infinity for m–uniform measures. For the reader's convenience we state again Proposition 6.16 below.

PROPOSITION 7.1. *If $\mu \in \mathcal{U}^m(\mathbb{R}^n)$, then there exists $\zeta \in \mathcal{U}^m(\mathbb{R}^n)$ such that $\mathrm{Tan}_m(\mu,\infty) = \{\zeta\}$.*

A first easy remark, which will be used many times in subsequent chapters, is that the condition of m–uniformity of the measure μ allows us to compute $\int \varphi d\mu$ for radial φ's, without any further information on μ. This is stated more precisely in the following lemma.

LEMMA 7.2. *Let $\varphi : \mathbb{R} \to \mathbb{R}^+$ be a Borel function, μ an m–uniform measure and y a point in the support of μ. Then*

$$\int_{\mathbb{R}^n} \varphi(|x|)\, d\mu(x) \;=\; \int_{\mathbb{R}^n} \varphi(|x-y|)\, d\mu(x) \;=\; \int_{\mathbb{R}^m} \varphi(|z|)\, d\mathscr{L}^m(z)\,. \tag{7.1}$$

PROOF. Denote by $B_r(y)$ the n–dimensional ball of radius r centered at $y \in \mathbb{R}^n$ and by $\tilde{B}_r(z)$ the m–dimensional ball of radius r centered at $z \in \mathbb{R}^m$. Since $\mu(B_r(0)) = \mu(B_r(y)) = \omega_m r^m = \mathscr{L}^m(\tilde{B}_r(0))$, the identity (7.1) is clear when φ is piecewise constant. Therefore, a standard density argument gives (7.1) in the general case. $\square$

Next we introduce a normalization of the measures $r^{-m}\mu_{0,r}$: Namely we multiply them by a Gaussian.

DEFINITION 7.3. *Let $\mu \in \mathcal{U}^m(\mathbb{R}^n)$. Then we set $\mu_r := r^{-m}e^{-|\cdot|^2}\mu_{0,r}$, that is, for every Borel function we have*

$$\int \varphi(x) d\mu_r(x) \;=\; r^{-m} \int e^{-\frac{|x|^2}{r^2}} \varphi\left(\frac{x}{r}\right)\, d\mu(x)\,.$$

Note that if $\nu \in \mathrm{Tan}_m(\mu, x)$ and $r_i \uparrow \infty$ is a sequence such that

$$r_i^{-m}\mu_{0,r_i} \;\stackrel{*}{\rightharpoonup}\; \nu\,,$$

then $\mu_{r_i} \stackrel{*}{\rightharpoonup} e^{-|\cdot|^2}\nu$. Therefore, the tangent measure to μ at infinity is unique if and only if the measures μ_r have a unique limit for $r \uparrow \infty$.

Moments. Since $\mu \in \mathcal{U}^m(\mathbb{R}^n)$, it is not difficult to check that for every polynomial P the function

$$F_P(r) \;:=\; \int P(z)\, d\mu_r(z)$$

is uniformly bounded. Assume we could prove the existence of the following limit for every polynomial P:

$$\lim_{r\uparrow\infty} F_P(r)\,. \tag{7.2}$$

We then would have that, if ζ and ξ are tangent measures at infinity to μ, then

$$\int e^{-|z|^2} P(z)\, d\zeta(z) \;=\; \int e^{-|z|^2} P(z)\, d\xi(z)$$

for every polynomial P. This is enough to conclude that the measures ζ and ξ coincide.

Therefore, our goal is to prove the existence of the limits (7.2). In order to do this we introduce the following notation.

Definition 7.4 (Moments). *Let $\mu \in \mathcal{U}^m(\mathbb{R}^n)$, $u_1, \ldots, u_k \in \mathbb{R}^n$, and $s \in \mathbb{R}^+$. Then we define*

$$I(s) \;:=\; \int e^{-s|z|^2}\, d\mu(z)$$

$$b^\mu_{k,s}(u_1,\ldots,u_k) \;:=\; \frac{(2s)^k}{k!} I(s)^{-1} \int \langle z, u_1\rangle\langle z, u_2\rangle \ldots \langle z, u_k\rangle\, e^{-s|z|^2}\, d\mu(z)\,.$$

The reader will recognize that $b^\mu_{k,s}(u_1,\ldots,u_k)$ is closely related to $F_P(s^{-1/2})$ when we choose $P(z) = \langle z, u_1\rangle\langle z, u_2\rangle \ldots \langle z, u_k\rangle$. Since, for each fixed k, the space of k–linear forms on $\mathbb{R}^n$ is a real vector space of finite dimension, we consider on it the standard topology. Under this convention, the limits

$$\lim_{s\downarrow 0} s^{-N/2} b^\mu_{N,s} \tag{7.3}$$

exist if and only if (7.2) exists. Hence, our final goal is the following:

Proposition 7.5. *If $\mu \in \mathcal{U}^m(\mathbb{R}^n)$, then the limits* (7.3) *exist.*

The moments $b^\mu_{N,s}$ are, in a certain sense, generalizations of the barycenter

$$b_r(\mu) \;=\; r^{-m} \int_{B_r(0)} z\, d\mu(z)\,,$$

defined in (3.17) and used in Section 3 to study α–uniform measures. One sees immediately the convenience of multiplying by a Gaussian, which allows to integrate over the whole $\mathbb{R}^n$. However, we will see soon that this is not the only reason for choosing the Gaussian: This choice will play an important role in many algebraic computations.

Note that, thanks to Lemma 7.2, $I(s)$ is independent of the measure μ. This is not the case for the k–linear forms $b^\mu_{k,s}$. However we will drop the superscript when the measure μ is clear from the context.

Taylor expansion. In Lemma 7.6 below, we will make use of the following notation

$$b^\mu_{k,s}(x^k) \;:=\; b^\mu_{k,s}(x, x, \ldots, x)\,.$$

A similar notation will be used whenever we will deal with k–linear forms. The existence of the limits (7.3) follow from a key calculation involving moments, stated in point (b) of the following lemma.

Lemma 7.6 (Taylor expansion 1). *Let $\mu \in \mathcal{U}^m(\mathbb{R}^n)$. Then*

(a) *There exists a dimensional constant C (depending only on m) such that*

$$\big|b_{k,s}(u_1,\dots,u_k)\big| \;\le\; C\frac{2^k k^{k/2}}{k!}s^{k/2}|u_1|\dots|u_k|\,. \tag{7.4}$$

(b) *For every $q\in\mathbb{N}$ there exists a constant C such that*

$$\left|\sum_{k=1}^{2q} b_{k,s}(x^k)-\sum_{k=1}^{q}\frac{s^k|x|^{2k}}{k!}\right| \;\le\; C\big(s|x|^2\big)^{q+\frac12} \qquad \textit{for every } x\in\operatorname{supp}(\mu). \tag{7.5}$$

Let us adopt the convention $b_{0,s}(x^0)=1$. Then, for $s|x|^2<1$, point (b) can be formally rewritten as

$$\sum_{k=0}^{\infty} b_{k,s}(x^k) \;=\; e^{s|x|^2} \qquad \text{for every } x\in\operatorname{supp}(\mu). \tag{7.6}$$

What follows is the "formal" computation that leads to (7.6), which will be rigorously justified in Section 2:

$$\begin{aligned}
\sum_{k=0}^{\infty} b_{k,s}(x^k) &= \sum_{k=0}^{\infty} I(s)^{-1}\int \frac{(2s\langle z,x\rangle)^k}{k!}e^{-s|z|^2}\,d\mu(z)\\
&= I(s)^{-1}\int\left[\sum_{k=0}^{\infty}\frac{(2s\langle z,x\rangle)^k}{k!}\right]e^{-s|z|^2}\,d\mu(z) \;=\; I(s)^{-1}\int e^{2s\langle z,x\rangle-s|z|^2}\,d\mu(z)\\
&= I(s)^{-1}e^{s|x|^2}\int e^{-s|x|^2+2s\langle z,x\rangle-s|z|^2}\,d\mu(z) \;=\; e^{s|x|^2}I(s)^{-1}\int e^{-s|z-x|^2}\,d\mu(z)\,.
\end{aligned}$$

Actually this computation turns out to be valid for *every* $x\in\mathbb{R}^n$. When in addition we have $x\in\operatorname{supp}(\mu)$, Lemma 7.2 gives $\int e^{-s|z-x|^2}\,d\mu(z)=\int e^{-s|z|^2}\,d\mu(z)=I(s)$, and hence (7.6). Point (b) of Lemma 7.6 is the starting point of next proposition.

Proposition 7.7 (Taylor expansion 2). *Let $\mu\in\mathcal{U}^m(\mathbb{R}^n)$. Then for $j,k\in\mathbb{N}$ there exist symmetric k–linear forms $b_k^{(j)}$ such that:*

$$\textit{For all } q\in\mathbb{N} \textit{ we have} \qquad b^{\mu}_{k,s} \;=\; \sum_{j=1}^{q}\frac{s^j b_k^{(j)}}{j!}+o(s^q)\,; \tag{7.7}$$

$$b_k^{(j)} \;=\; 0 \qquad \textit{if } k>2j\,; \tag{7.8}$$

$$\sum_{k=1}^{2q} b_k^{(q)}(x^k) \;=\; |x|^{2q} \qquad \textit{for all } q\in\mathbb{N} \textit{ and all } x\in\operatorname{supp}(\mu). \tag{7.9}$$

This proposition concludes the proof of the uniqueness of the tangent measure at infinity. Indeed, according to all that has been discussed so far, we just need to show the existence of the limit (7.3), which is a trivial consequence of (7.7) and (7.8).

When we have to specify the dependence of the form $b_k^{(j)}$ on the measure μ we will write $b_k^{\mu,(j)}$. In order to visualize the relation between (7.6) and (7.9), we will use the table below. Clearly, the first row gives the Taylor expansion of $e^{s|x|^2}-1$. When $x\in\operatorname{supp}(\mu)$, the same is true for the first column, according to (7.6). Moreover, according to (7.7) and (7.8), the interior rows are the expansions of $b_{k,s}(x^k)$. Therefore, the interior columns must be "expansions" of $s^k|x|^{2k}/k!$ when $x \in \operatorname{supp}(\mu)$.

$$
\begin{array}{ccccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & & & & & \\
b_{5,s}(x^5) & 0 & 0 & \frac{s^3}{3!}b_5^{(3)}(x^5) & \frac{s^4}{4!}b_5^{(4)}(x^5) & \frac{s^5}{5!}b_5^{(5)}(x^5) & \cdots \\
b_{4,s}(x^4) & 0 & \frac{s^2}{2!}b_4^{(2)}(x^4) & \frac{s^3}{3!}b_4^{(3)}(x^4) & \frac{s^4}{4!}b_4^{(4)}(x^4) & \frac{s^5}{5!}b_4^{(5)}(x^4) & \cdots \\
b_{3,s}(x^3) & 0 & \frac{s^2}{2!}b_3^{(2)}(x^3) & \frac{s^3}{3!}b_3^{(3)}(x^3) & \frac{s^4}{4!}b_3^{(4)}(x^3) & \frac{s^5}{5!}b_3^{(5)}(x^3) & \cdots \\
b_{2,s}(x^2) & sb_2^{(1)}(x^2) & \frac{s^2}{2!}b_2^{(2)}(x^2) & \frac{s^3}{3!}b_2^{(3)}(x^2) & \frac{s^4}{4!}b_2^{(4)}(x^2) & \frac{s^5}{5!}b_2^{(5)}(x^2) & \cdots \\
b_{1,s}(x) & sb_1^{(1)}(x) & \frac{s^2}{2!}b_1^{(2)}(x) & \frac{s^3}{3!}b_1^{(3)}(x) & \frac{s^4}{4!}b_1^{(4)}(x) & \frac{s^5}{5!}b_1^{(5)}(x) & \cdots \\
 & & & & & & \\
e^{s|x|^2}-1 & s|x|^2 & \frac{s^2}{2!}|x|^4 & \frac{s^3}{3!}|x|^6 & \frac{s^4}{4!}|x|^8 & \frac{s^5}{5!}|x|^{10} & \cdots
\end{array}
$$

Plan of the chapter. In section 1 we will show how Proposition 7.7 implies Proposition 7.1. In section 2 we will prove Lemma 7.6. Finally, in section 3 we will use Lemma 7.6 to prove Proposition 7.7.

1. From Proposition 7.7 to the uniqueness of the tangent measure at infinity

From Proposition 7.7 to Proposition 7.1. From Proposition 7.7 we observe that:

- If N is odd, then
$$\lim_{s\downarrow 0}\frac{b_{N,s}(x_1,\dots,x_N)}{s^{N/2}} \;=\; 0$$
- For any $k\in\mathbb{N}$
$$\lim_{s\downarrow 0}\frac{b_{2k,s}(x_1,\dots,x_{2k})}{s^k} \;=\; b_{2k}^{(k)}(x_1,\dots,x_{2k})\,.$$

In both cases we conclude that the limits
$$\lim_{s\downarrow 0}\frac{b_{N,s}(x_1,\dots,x_N)}{s^{N/2}}$$
exist. Recall that
$$b_{N,s}(u_1,\dots,u_N) \;=\; \frac{(2s)^N}{N!}I(s)^{-1}\int\langle z,u_1\rangle\langle z,u_2\rangle\dots\langle z,u_N\rangle e^{-s|z|^2}\,d\mu(z)\;,$$
where
$$I(s) \;=\; \int e^{-s|z|^2}\,d\mu(z)\,.$$

From Lemma 7.2 and Proposition B.1 we have

$$I(s) \;=\; \int e^{-s|z|^2}\, d\mathscr{L}^m(z) \;=\; \frac{1}{s^{m/2}} \int e^{-|z|^2}\, d\mathscr{L}^m(z) \;=\; \left(\frac{\pi}{s}\right)^{m/2} .$$

Therefore

$$\frac{b_{N,s}(u_1,\ldots,u_N)}{s^{N/2}} \;=\; C(N,m) s^{N/2-m/2} \int \langle z,u_1\rangle\langle z,u_2\rangle\ldots\langle z,u_N\rangle e^{-s|z|^2}\, d\mu(z)\,,$$

where $C(N,m)$ is a positive dimensional constant, independent of s. If we define $r := s^{1/2}$ we obtain

$$\begin{aligned}
\frac{b_{N,s}(u_1,\ldots,u_N)}{s^{N/2}} &= \frac{C(N,m)r^N}{r^m} \int \langle z,u_1\rangle\ldots\langle z,u_N\rangle e^{-r^2|z|^2}\, d\mu(z)\\
&= \frac{C(N,m)}{r^m} \int \langle rz,u\rangle\ldots\langle rz,u_N\rangle e^{-|rz|^2}\, d\mu(z)\\
&= C(N,m) \int \langle x,u_1\rangle\ldots\langle x,u_N\rangle e^{-|x|^2}\, d\left[\frac{\mu_{0,r}}{r^m}\right](x)\,.
\end{aligned}$$

Therefore, we conclude that the limits (7.2) exist whenever P is a polynomial of the form $\langle u_1,\cdot\rangle\ldots\langle u_N,\cdot\rangle$.

Let $\{r_k\}$ and $\{s_k\}$ be two sequences of real numbers such that

- $r_k \uparrow \infty$, $s_k \uparrow \infty$;
- $r_k^{-m}\mu_{x,r_k} \overset{*}{\rightharpoonup} \nu^1$, $s_k^{-m}\mu_{x,s_k} \overset{*}{\rightharpoonup} \nu^2$.

We set $\tilde\nu^1 := e^{-|\cdot|^2}\nu^1$ and $\tilde\nu^2 := e^{-|\cdot|^2}\nu^2$. Clearly,

$$\mu_{r_k} \overset{*}{\rightharpoonup} \tilde\nu^1 \qquad \mu_{s_k} \overset{*}{\rightharpoonup} \tilde\nu^2\,.$$

Note that for any $j\in\mathbb{N}$ and $\varepsilon>0$ there exists $M>0$ such that

$$\int_{\mathbb{R}^n\setminus B_M(0)} |u|^j d\mu_r(z) \;\le\; \varepsilon\,.$$

Therefore, we conclude that

$$\begin{aligned}
\lim_{r_k\downarrow 0} \int \langle z,u\rangle^j\, d\mu_{r_k}(z) &= \int \langle z,u\rangle^j\, d\tilde\nu^1(z)\\
\lim_{s_k\downarrow 0} \int \langle z,u\rangle^j\, d\mu_{s_k}(z) &= \int \langle z,u\rangle^j\, d\tilde\nu^2(z)\,.
\end{aligned}$$

Hence, following the previous discussion, we conclude that

$$\int \langle z,u\rangle^j\, d\tilde\nu^1(z) \;=\; \int \langle z,u\rangle^j\, d\tilde\nu^2(z)\,. \tag{7.10}$$

This implies that for every polynomial P in n variables we have

$$\int e^{-|z|^2} P(z)\, d\nu^1(z) \;=\; \int e^{-|z|^2} P(z)\, d\nu^2(z)\,. \tag{7.11}$$

Using the expansion

$$e^{-a|z|^2} \;=\; \sum_{i=0}^{\infty} \frac{\left(-\sqrt{a}|z|^2\right)^i}{i!}\,,$$

one also obtains the equality

$$\int e^{-(1+a)|z|^2} P(z)\, d\nu^1(z) \;=\; \int e^{-(1+a)|z|^2} P(z)\, d\nu^2(z) \tag{7.12}$$

for every nonnegative a. Therefore, a density argument like that used in Step 2 of the proof of Proposition 6.11 (see also Lemma 6.14) gives $\nu^1 = \nu^2$. We include it below for the reader's convenience.

Clearly it suffices to show

$$\int \varphi(z)\, d\nu^1(z) \;=\; \int \varphi(z)\, d\nu^2(z) \tag{7.13}$$

for every $\varphi \in C_c(\mathbb{R}^n)$. Let $\mathcal{B}$ be the vector space generated by functions of the form

$$b + e^{-(1+a)|z|^2} P(z)\,,$$

where $a \geq 0$, $b \in \mathbb{R}$ and P is a polynomial. To prove (7.13), it suffices to show the following:

(D) For every compactly supported function $\psi \in C(\mathbb{R}^n)$ there exists a sequence $\{\psi_i\} \subset \mathcal{B}$ which converges uniformly to ψ.

Indeed assume (D), fix $\varphi \in C_c(\mathbb{R}^n)$, and choose $\{\psi_i\} \subset \mathcal{B}$ which converges uniformly to $\psi := e^{|\cdot|^2}\varphi$. Then we have

$$\int e^{-|z|^2} \psi_i(z)\, d\nu^1(z) \;=\; \int e^{-|z|^2} \psi_i(z)\, d\nu^2(z)\,. \tag{7.14}$$

Since $\{\psi_i\}$ is uniformly bounded, we let $i \uparrow \infty$ in (7.14) to obtain (7.13).

In order to show (D), fix $\psi \in C_c(\mathbb{R}^n)$, denote by $\mathbb{S}^n$ the usual one–point compactification of $\mathbb{R}^n$, and denote by $\tilde{\psi} \in C_c(\mathbb{S}^n)$ the unique continuous extension of ψ. For every $\chi \in \mathcal{B}$ there exists as well a unique extension $\tilde{\chi} \in C_c(\mathbb{S}^n)$. Denote by $\tilde{\mathcal{B}}$ the vector space of such extensions. Then $\tilde{\mathcal{B}}$ is an algebra of continuous functions on a compact space, it separates the points, and it vanishes at no point. Therefore, we can apply the Stone–Weierstrass Theorem to conclude that there exists a sequence $\{\tilde{\psi}_i\} \subset \tilde{\mathcal{B}}$ which converges uniformly to $\tilde{\psi}$. The corresponding sequence $\{\psi_i\} \in \mathcal{B}$ also converges uniformly to ψ. This concludes the proof of (D) and of the proposition. □

2. Elementary bounds on $b_{k,s}$ and the expansion (7.5)

Proof of Lemma 7.6. **(a)** Recall that

$$b_{k,s}(u_1, \ldots, u_k) \;=\; \frac{(2s)^k}{k!} I(s)^{-1} \int \langle z, u_1\rangle \ldots \langle z, u_k\rangle e^{-s|z|^2}\, d\mu(z)\,.$$

Hence, we obtain

$$\big|b_{k,s}(u_1, \ldots, u_k)\big| \;\leq\; |u_1| \ldots |u_k| \frac{(2s)^k}{k!} I(s)^{-1} \int |z|^k e^{-s|z|^2} d\mu(z)\,.$$

Recall the computation already performed in the proof of Proposition 7.1: From Lemma 7.2 and Proposition B.1 we have

$$
\begin{aligned}
I(s) \;&:=\; \int e^{-s|z|^2}\, d\mu(z) \;=\; \int e^{-s|z|^2} d\mathscr{L}^m(z) \\
&=\; s^{-m/2}\int e^{-|z|^2}\, d\mathscr{L}^m(z) \;=\; \left(\frac{\pi}{s}\right)^{m/2} .
\end{aligned}
\tag{7.15}
$$

Therefore

$$
\frac{(2s)^k}{k!} I(s)^{-1} \int |z|^k e^{-s|z|^2} d\mu(z) \;=\; \frac{2^k}{\pi^{m/2} k!} s^{k/2} \int |s^{1/2} z|^k e^{-|s^{1/2} z|^2}\, d\big[s^{m/2}\mu(z)\big] . \tag{7.16}
$$

Using Lemma 7.2 and changing variables we obtain

$$
\begin{aligned}
\int |s^{1/2} z|^k e^{-|s^{1/2} z|^2}\, d\big[s^{m/2}\mu(z)\big] \;&=\; \int |s^{1/2} z|^k e^{-|s^{1/2} z|^2}\, d\big[s^{m/2}\mathscr{L}^m(z)\big] \\
&=\; \int |z|^k e^{-|z|^2}\, d\mathscr{L}^m(z) .
\end{aligned}
\tag{7.17}
$$

From (B.4) and (B.6) of Proposition B.1 we obtain

$$
\int |z|^k e^{-|z|^2}\, d\mathscr{L}^m(z) \;\leq\; C(m) k^{k/2} , \tag{7.18}
$$

where $C(m)$ is a dimensional constant that depends only on m.

Hence, (7.15), (7.16), (7.17), and (7.18) give the bound (7.4).

(b) When $s|x|^2 \geq 1$, we can use the following rough bounds:

$$
\begin{aligned}
\left|\sum_{k=1}^{2q} b_{k,s}(x^k) - \sum_{k=1}^{q} \frac{s^k |x|^{2k}}{k!}\right| \;&\leq\; \left|\sum_{k=1}^{2q} b_{k,s}(x^k)\right| + \left|\sum_{k=1}^{q} \frac{s^k |x|^{2k}}{k!}\right| \\
&\overset{(7.4)}{\leq}\; C(m)\big(s|x|^2\big)^{q+1/2} \sum_{k=1}^{\infty} \frac{2^k k^{k/2}}{k!} + \big(s|x|^2\big)^{q+1/2} \sum_{k=1}^{\infty} \frac{1}{k!} \\
&\leq\; C_1(m)\big(s|x|^2\big)^{q+1/2} ,
\end{aligned}
$$

where here we have used the summability of the series

$$
\sum_{k=1}^{\infty} \frac{2^k k^{k/2}}{k!} ,
$$

which follows from Stirling's formula $k! \geq C k^k e^{-k}$. Note that in this case we do not need the condition $x \in \operatorname{supp}(\mu)$. On the other hand, such a condition is crucial when $s|x|^2 < 1$.

First of all, note that

$$
\sum_{k=0}^{\infty} \frac{s^k |x|^{2k}}{k!} \;=\; e^{s|x|^2} .
$$

More precisely:

$$\begin{aligned}\left|e^{s|x|^2} - \sum_{k=0}^{q} \frac{s^k|x|^{2k}}{k!}\right| &= \sum_{k=q+1}^{\infty} \frac{s^k|x|^{2k}}{k!} \\ &\leq s^{q+1}|x|^{2q+2} \sum_{k=0}^{\infty} \frac{1}{k!} = es^{q+1}|x|^{2q+2}\,. \end{aligned} \tag{7.19}$$

By the bounds (7.4) it turns out that, for $s|x|^2 < 1$,

$$\sum_{k=1}^{\infty} \left|b_{k,s}(x^k)\right| \leq \sum_{k=1}^{\infty} C\frac{2^k k^{k/2}}{k!} s^{k/2}|x|^k \leq C\sum_{k=1}^{\infty} \frac{2^k k^{k/2}}{k!}\,.$$

We already observed in the previous step that the last series is summable. Therefore, we conclude that

$$\sum_{k=1}^{\infty} b_{k,s}(x^k)$$

is summable for $s|x|^2 < 1$. Moreover, we can estimate

$$\begin{aligned}\left|\sum_{k=1}^{\infty} b_{k,s}(x^k) - \sum_{k=1}^{2q} b_{k,s}(x^k)\right| &\leq \sum_{k=2q+1}^{\infty} \left|b_{k,s}(x^k)\right| \leq \sum_{k=2q+1}^{\infty} \frac{2^k k^{k/2}}{k!} s^{k/2}|x|^k \\ &\leq (s|x|^2)^{q+1/2} \sum_{k=1}^{\infty} \frac{2^k k^{k/2}}{k!} \leq C(s|x|^2)^{q+1/2}\,. \end{aligned} \tag{7.20}$$

Let us fix the convention that $b_{0,s}(x^0) := 1$. Then, from (7.19) and (7.20) it follows that the proof is complete provided we show the equality

$$\sum_{k=0}^{\infty} b_{k,s}(x^k) = e^{s|x|^2} \tag{7.21}$$

for every $s \in \mathbb{R}^+$ and $x \in \mathbb{R}^n$ such that $s|x|^2 < 1$ and $x \in \operatorname{supp}(\mu)$.

From (7.20) and from the definition of $b_{k,s}$ we have

$$\sum_{k=0}^{\infty} b_{k,s}(x^k) = \lim_{q\uparrow\infty} \sum_{k=0}^{q} I(s)^{-1} \int \frac{(2s\langle z, x\rangle)^k}{k!} e^{-s|z|^2}\, d\mu(z)\,.$$

Note that

$$\left|\sum_{k=0}^{q} \frac{(2s\langle z, x\rangle)^k}{k!} e^{-s|z|^2}\right| \leq e^{-s|z|^2} \sum_{k=0}^{q} \frac{(2s|z||x|)^k}{k!} \leq e^{-s(|z|^2+2|z||x|)}\,.$$

Since $f(\cdot) = e^{-s(|\cdot|^2+2|\cdot||x|)} \in L^1(\mu)$, by the Dominated Convergence Theorem we conclude

$$\begin{aligned}
\sum_{k=0}^{\infty} b_{k,s}(x^k) &= I(s)^{-1} \int \left[\sum_{k=0}^{\infty} \frac{(2s\langle z, x\rangle)^k}{k!} \right] e^{-s|z|^2}\, d\mu(z) \;=\; I(s)^{-1} \int e^{2s\langle z,x\rangle - s|z|^2}\, d\mu(z) \\
&= I(s)^{-1} e^{s|x|^2} \int e^{-s|x|^2+2s\langle z,x\rangle - s|z|^2}\, d\mu(z) \\
&= e^{s|x|^2} I(s)^{-1} \int e^{-s|z-x|^2}\, d\mu(z)\,.
\end{aligned} \tag{7.22}$$

Since $x \in \operatorname{supp}(\mu)$, from Lemma 7.2 we obtain

$$\int e^{-s|z-x|^2}\, d\mu(z) \;=\; \int e^{-s|z|^2}\, d\mu(z) \;=\; I(s)\,.$$

Hence, from (7.22) we conclude (7.21). This completes the proof. $\square$

3. Proof of Proposition 7.7

Before coming to the proof of Proposition 7.7, we introduce some notation.

Definition 7.8. *We denote by $\bigodot^k \mathbb{R}^n$ the vector space of symmetric k–tensors on $\mathbb{R}^n$. When $u_1, \ldots, u_k \in \mathbb{R}^n$ we denote by $u_1 \odot \ldots \odot u_k$ their symmetric tensor product, that is, the tensor*

$$\frac{1}{k!} \sum_{\sigma \in G_k} u_{\sigma(1)} \otimes \ldots \otimes u_{\sigma(k)}\,,$$

where G_k denotes the group of permutations of $\{1, \ldots, k\}$. We use the shorthand u^k when $u_1 = \ldots = u_k = u$.

For each s we can regard $b_{k,s}$ as an element of $\operatorname{Hom}(\bigodot^k \mathbb{R}^n, \mathbb{R})$ and therefore we consider the map $s \to b_{k,s}$ as a curve in $\operatorname{Hom}(\bigodot^k \mathbb{R}^n, \mathbb{R})$.

Definition 7.9. *For every pair of positive integers k, n we define $X^{k,n}$ as the direct sum*

$$\mathbb{R}^n \oplus \bigodot^2 \mathbb{R}^n \oplus \ldots \oplus \bigodot^k \mathbb{R}^n\,.$$

We denote by P_j the canonical projection of $X^{k,n}$ on $\bigodot^j \mathbb{R}^n$.

We can extend $b_{k,s}$ to a linear functional on $X^{2q,n}$ by setting $b_{k,s}|_{\bigodot^j \mathbb{R}^n} = 0$ for every $j \neq k$. Therefore, the map $s \to \sum_{k=1}^{2q} b_{k,s}$ can be considered as a curve in $\operatorname{Hom}(X^{2q,n}, \mathbb{R})$.

Remark 7.10. *On every $\bigodot^k \mathbb{R}^n$ there exists a unique scalar product $\langle \cdot, \cdot \rangle_k$ such that*

$$\langle u_1 \odot \ldots \odot u_k, v_1 \odot \ldots \odot v_k \rangle_k \;=\; \langle u_1, v_1\rangle \ldots \langle u_k, v_k\rangle\,.$$

Definition 7.11. *Let k and n be positive integers. We define on $X^{k,n}$ the scalar product $\langle\langle\, \cdot, \cdot \rangle\rangle$ as*

$$\langle\langle\, u, v \rangle\rangle \;:=\; \sum_{j=1}^{k} \frac{2^j \langle P_j(u), P_j(v)\rangle_j}{j!}\,.$$

Moreover, we set $\|u\| := \langle\langle\, u, u\rangle\rangle$ and for every linear subspace $V \subset X$ we denote by $V^\perp$ the orthogonal subspace

$$V^\perp \;:=\; \big\{ x \in X \;:\; \langle\langle\, x, v\rangle\rangle = 0 \qquad \forall v \in V \big\}\,.$$

We are now ready to give a brief outline of the proof of Proposition 7.7. The core of this Proposition is the Taylor expansion (7.7). Let $q \in \mathbb{N}$. Roughly speaking (7.5) determines (up to order q) the Taylor expansion of the function $f(s) = \sum_{i=1}^{2q} b_{k,s}(x^k)$ when $x \in \operatorname{supp}(\mu)$. Here it is convenient to introduce the multilinear notation, in order to consider the map

$$s \;\rightarrow\; b_s = \sum_{k=1}^{2q} b_{k,s} \in \operatorname{Hom}\left(X^{2q,n}, \mathbb{R}\right).$$

as a curve of linear operators acting on the tensor space $X = X^{2q,n}$. Therefore, (7.5) gives the expansion

$$b_s(x + x^2 + \ldots + x^{2q}) \;=\; \sum_{k=1}^{q} \frac{s^k |x|^{2k}}{k!} + \|x\|^{2q+1} o(s^q) \tag{7.23}$$

whenever x belongs to $\operatorname{supp}(\mu)$. By linearity, this determines the values of b_s on the vector space V generated by $\{x + x^2 + \ldots + x^{2q} \;:\; x \in \operatorname{supp}(\mu)\}$. The tensor notation gives a concise way to express this. Indeed it is not difficult to see that there exists a unique smooth curve

$$[0, \infty[\ni s \;\rightarrow\; \omega_s \in \operatorname{Hom}(X, \mathbb{R})$$

such that $\omega_s(y^j) = 0$ if j is odd and $\omega_s(y^{2k}) = s^k |y|^{2k}/k!$. Therefore, (7.23) can be written as $b_s|_V = \omega_s|_V + o(s^q)$.

Another key remark is that, for each s, there exists a subspace F_s such that the linear functional b_s vanishes on F_s and $V \oplus F_s = X$. Such an F_s is given by an explicit formula in (7.28). If we denote by Q_s the projection of X over V along F_s, then we have $b_s = b_s|_V \circ Q_s = \omega_s \circ Q_s + o(s^q)$. Hence, we just need to show that the curve of operators $s \rightarrow Q_s$ has a Taylor expansion around $s = 0$. Indeed, using the formula (7.28), we will show that this curve is analytic in a neighborhood of 0.

Proof of Proposition 7.7. First of all, note that (7.8) follows directly from (7.7) and point (a) of Lemma 7.6. Similarly (7.9) follows as well from (7.7) and point (b) of Lemma 7.6. Indeed fix $q \in \mathbb{N}$. From (7.5) we have

$$\left| \sum_{k=1}^{2q} b_{k,s}(x^k) - \sum_{k=1}^{q} \frac{s^k |x|^{2k}}{k!} \right| \;\leq\; C\big(s|x|^2\big)^{q+\frac{1}{2}} \qquad \text{for every } x \in \operatorname{supp}(\mu). \tag{7.24}$$

From (7.7) we have

$$b_{k,s}(x^k) \;=\; \sum_{j=1}^{q} \frac{s^j b_k^{(j)}(x^k)}{j!} + o(s^q) \tag{7.25}$$

for every $x \in \mathbb{R}^n$. Therefore, for any fixed $x \in \operatorname{supp}(\mu)$, recalling (7.8) we can write

$$\begin{aligned} \Bigg| s\Big(b_1^{(1)}(x) + b_2^{(1)}(x^2) - |x|^2\Big) + \frac{s^2}{2!}\Big(b_1^{(2)}(x) + b_2^{(2)}(x^2) + b_3^{(2)}(x^3) + b_4^{(2)}(x^4)\Big) \\ + \ldots + \frac{s^q}{q!}\left(\sum_{i=1}^{2q} b_i^{(q)}(x^i) - |x|^{2q}\right) \Bigg| \;=\; o(s^q)\,. \end{aligned} \tag{7.26}$$

For $q = 1$ we have

$$\left| b_1^{(1)}(x) + b_2^{(1)}(x^2) - |x|^2 \right| \;=\; s^{-1}o(s)$$

and hence $b_1^{(1)}(x) + b_2^{(1)}(x^2) = 0$. By induction we then obtain

$$\left| \sum_{i=1}^{2q} b_i^{(q)}(x^i) - |x|^{2q} \right| \;=\; s^{-q}o(s^q)\,,$$

and hence

$$|x|^{2q} \;=\; \sum_{i=1}^{2q} b_i^{(q)}(x^i)\,.$$

It remains to prove point (7.7).

Proof of (7.7) Let us fix $q \in \mathbb{N}$ and consider the curve

$$\mathbb{R}^+ \ni s \quad \rightarrow \quad b_s \;:=\; \sum_{k=1}^{2q} b_{k,s} \in \mathrm{Hom}\,(X^{2q,n}, \mathbb{R})\,.$$

For simplicity we will drop the superscripts from $X^{2q,n}$.

For any $k \in \mathbb{N}$ we denote by $\hat{w}_{2k}$ the element of $\mathrm{Hom}\,(X, \mathbb{R})$ such that

$$\begin{aligned}
\hat{w}_{2k}(y) \;&=\; 0 \qquad \text{for every } y \in \textstyle\bigodot^j \mathbb{R}^n \text{ with } j \neq 2k \\
\hat{w}_{2k}(x^{2k}) \;&=\; \frac{|x|^{2k}}{k!}\,.
\end{aligned}$$

For every $s \in \mathbb{R}^+$ we denote by ω_s the element of $\mathrm{Hom}\,(X, \mathbb{R})$ given by

$$\omega_s \;:=\; \sum_{k=1}^{q} s^k \hat{w}_{2k}\,.$$

Finally, let V be the linear subspace of X generated by the elements

$$x + x^2 + \ldots + x^{2q} \;=\; x + x \odot x + \ldots + x \odot \ldots \odot x \qquad \text{for } x \in \mathrm{supp}\,(\mu).$$

The aim of this new notation is to rewrite the formula (7.5) as

$$b_s(y) \;=\; \omega_s(y) + \|y\|^{2q+1} o(s^q) \qquad \text{for every } y \in V\,. \tag{7.27}$$

Now for every $s \in \mathbb{R}^+$ we define the subspace $F_s \subset X$ as

$$F_s \;:=\; \left\{ u \in X \;:\; \left\langle\!\!\left\langle \sum_{k=1}^{2q} s^k P_k(u), v \right\rangle\!\!\right\rangle \;=\; 0 \quad \forall v \in V \right\}. \tag{7.28}$$

Clearly $F_s \oplus V = X$, because the bilinear form

$$a_s(u, w) \;:=\; \left\langle\!\!\left\langle \sum_{k=1}^{2q} s^k P_k(u), w \right\rangle\!\!\right\rangle$$

is a scalar product on X.

Therefore, there exists a unique linear map $Q_s : X \to X$ such that:

$$\begin{cases} Q_s(v) = v & \text{for every } v \in V \\ \\ Q_s(v) = 0 & \text{for every } v \in F_s. \end{cases}$$

Clearly (7.27) yields

$$b_s(u) \;=\; \omega_s(Q_s(u)) + \|u\|^{2q+1} o(s^q) \qquad \text{for } u \in V. \tag{7.29}$$

On the other hand, from the definition of b_s, it follows that

$$b_s(u) \;=\; \int_{\mathbb{R}^n} a_s(u, v + v^2 + \ldots + v^{2q})\, d\mu(v)\,.$$

Therefore, we have

$$b_s(u) \;=\; 0 \;=\; \omega_s(0) \;=\; \omega_s(Q_s(u)) \qquad \text{for every } u \in F_s. \tag{7.30}$$

By the linearity of b_s, (7.29) and (7.30) yield

$$b_s \;=\; \omega_s \circ Q_s + o(s^q)\,. \tag{7.31}$$

In (7.31) we understand b_s and ω_s as curves in $\mathrm{Hom}\,(X, \mathbb{R})$ and Q_s as a curve in $\mathrm{Hom}\,(X, X)$.

Note that ω_s can also be defined for $s = 0$ and it yields an analytic curve $[0, \infty[\ni s \to \omega_s$. Therefore (7.7) would be implied by the following claim:

$$\text{The curve }]0, \infty[\ni s \to Q_s \text{ can be extended analytically to } s = 0. \tag{7.32}$$

Analyticity of Q_s at $s = 0$. In what follows, for every vector space $A \subset X$, we will denote by P_A the orthogonal projection on A with respect to the scalar product $\langle\langle\cdot,\cdot\rangle\rangle$. Recall that P_j is the orthogonal projection on $\bigodot^j \mathbb{R}^n$.

Observe that

$$\left(\sum_{k=1}^{2q} s^k P_k\right) \circ \left(\sum_{j=1}^{2q} s^{-j} P_j\right) \;=\; \sum_{k,j} s^{k-j} P_k \circ P_j \;=\; \sum_{k=1}^{2q} P_k\,.$$

Since the last linear map is the identity, we conclude that $\sum_{j=1}^{2q} s^{-j} P_j$ is the inverse of $\sum_{k=1}^{2q} s^k P_k$. Therefore

$$x \in F_s \qquad \Longleftrightarrow \qquad x \in \left[\sum_{j=1}^{2q} s^{-j} P_j\right] (V^\perp)\,. \tag{7.33}$$

We decompose the linear space $V^\perp$ into a direct sum $\bigoplus_{k=1}^{2q} V_k$, where the linear spaces V_k are defined inductively as

$$\begin{aligned} V_1 \;&:=\; V^\perp \cap \textstyle\bigodot^1 \mathbb{R}^n \\ \\ V_2 \;&:=\; \left\{V^\perp \cap \left[\textstyle\bigodot^1 \mathbb{R}^n \oplus \textstyle\bigodot^2 \mathbb{R}^n\right]\right\} \cap V_1^\perp \\ \\ V_k \;&:=\; \left\{V^\perp \cap \left[\bigoplus_{j\le k} \textstyle\bigodot^j \mathbb{R}^n\right]\right\} \cap \bigcap_{j<k} V_j^\perp\,. \end{aligned}$$

Note that the sets V_k are pairwise orthogonal and they are all orthogonal to V. Define a linear map $A_s : X \to X$ in the following way:

- A_s is the identity on V;
- A_s on V_k is given by $P_k + sP_{k-1} + \ldots + s^{k-1}P_1$.

Note that A_s maps V into V and $V^\perp$ into F_s. Moreover, note that the curve $s \to A_s$ is analytic. We claim that A_0 is invertible. In order to show this, note that

- $\ker P_k \cap V_k = \{0\}$. Indeed, if $w \in V_k$ and $P_k(w) = 0$, then $w \in \bigoplus_{j<k} \bigodot^j \mathbb{R}^n$ and therefore $w \in V_{k-1}$; since $w \in V_k \subset V_{k-1}^\perp$, we conclude that $w = 0$.
- $P_j(V_k) = 0$ for $j > k$, since $V_k \subset \bigoplus_{i \le k} \bigodot^i \mathbb{R}^n$.
- $P_k(V_k) \cap V = \{0\}$. Indeed, if $x \in V_k$ and $P_k(x) \in V$, then $\langle\langle\, x, P_k(x)\rangle\rangle\ = 0$. Since $\langle\langle\, x, P_k(x)\rangle\rangle\ = |P_k(x)|^2$, we conclude that $P_k(x) = 0$.

These statements imply that the spaces $V, P_1(V_1), \ldots, P_{2q}(V_{2q})$ are pairwise transversal and

$$\dim(V) + \sum \dim(P_i(V_i)) \;=\; \dim(V) + \sum \dim(V_i) \;=\; \dim(X)\,.$$

Therefore, we conclude that A_0 is invertible.

Since A_s analytic, this implies that, in a neighborhood of 0, A_s is invertible and the map $s \to A_s^{-1}$ is analytic. Set $\tilde{Q}_s := P_V(A_s^{-1})$ and note that:

- $\tilde{Q}_s$ is analytic in a neighborhood of 0 because A_s^{-1} is analytic.
- $\tilde{Q}_s$ is the identity on V, since both P_V and A_s^{-1} are the identity on V.
- For $s > 0$ A_s^{-1} maps F_s into $V^\perp$, and therefore $\tilde{Q}_s = 0$ on F_s.

Hence, $Q_s = \tilde{Q}_s$ for $s > 0$, which implies that Q_s has an analytic extension at 0. $\square$

CHAPTER 8

Flat versus curved at infinity

The aim of this chapter is to prove Proposition 6.18. In particular we will prove the following stronger statement.

Proposition 8.1. *If $m = 0, 1, 2$ and $\mu \in \mathcal{U}^m(\mathbb{R}^n)$, then μ is flat at infinity. If $m \geq 3$, then there exists $\varepsilon > 0$ (which depends only on m and n) such that*

- *If λ is the tangent measure at infinity to $\mu \in \mathcal{U}^m(\mathbb{R}^n)$ and*

$$\min_{V \in G(m,n)} \int_{B_1(0)} \mathrm{dist}^2(x, V)\, d\lambda(x) \leq \varepsilon\,,$$

then λ is flat.

Remark 8.2. *Recall that the previous proposition is optimal in the following sense: For $n \geq 4$ and $m = n-1$ the measure $\mu = \mathscr{H}^m \llcorner \{x_1^2 + x_2^2 + x_3^2 = x_4^2\}$ is in $\mathcal{U}^m(\mathbb{R}^n)$ and clearly the tangent measure to μ at infinity is not flat (see Proposition 6.11 and Theorem 6.12).*

In the proof of this proposition, a key role is played by the information gained in the previous section: The uniqueness of the tangent measure at infinity. This uniqueness implies a "cone" property of the tangent measure at infinity. Indeed, let μ be a given measure and consider $\lambda \in \mathrm{Tan}_m(\mu, \infty)$. Let $r_i \uparrow \infty$ be some sequence such that $r_i^{-m}\mu_{0,r_i} \overset{*}{\rightharpoonup} \lambda$. Fix $\rho > 0$. Then $(\rho r_i)^{-m}\mu_{0,\rho r_i} \overset{*}{\rightharpoonup} \rho^{-m}\lambda_{0,\rho}$. Therefore, $\rho^{-m}\lambda_{0,\rho}$ belongs to $\mathrm{Tan}_m(\mu, \infty)$. If in addition the tangent measure to μ at infinity is unique, we conclude

$$\lambda_{0,\rho} \;=\; \rho^m \lambda \qquad \text{for every } \rho > 0. \tag{8.1}$$

It is not difficult to see that (8.1) implies the following:

$$x \in \mathrm{supp}\,(\lambda) \qquad \Longrightarrow \qquad \rho x \in \mathrm{supp}\,(\lambda) \quad \forall \rho > 0\,. \tag{8.2}$$

This consideration justifies the following definition.

Definition 8.3. *A measure λ which satisfies* (8.1) *is called a* conical measure*.*

We summarize the information gained so far in the following

Corollary 8.4 (Conical property of the tangent measure at infinity). *Let $\mu, \lambda \in \mathcal{U}^m(\mathbb{R}^n)$ be such that* $\mathrm{Tan}_m(\mu, \infty) = \{\lambda\}$*. Then λ is a conical measure and therefore satisfies* (8.1) *and* (8.2)*.*

Conical uniform measures. Proposition 8.1 holds because the same conclusion holds for *any* conical uniform measure:

Proposition 8.5. *Let $\lambda \in \mathcal{U}^m(\mathbb{R}^n)$ be conical. If $m \leq 2$, then λ is flat.*
When $m \geq 3$, then there exists $\varepsilon > 0$ (which depends only on m and n) such that, if

$$\min_{V \in G(m,n)} \int_{B_1(0)} \mathrm{dist}^2\,(x, V)\, d\lambda(x) \leq \varepsilon\,, \tag{8.3}$$

then λ is flat.

The combination of the conical and uniform properties yield many useful pieces of information on the tangent measure λ. In particular, if we fix x, a change of variables in the integrals that define the moments gives that

$$\text{the function } g(s) \;:=\; b^{\lambda}_{j,s}(x^j) \text{ is of the form } cs^{j/2}. \tag{8.4}$$

Therefore, from the Taylor expansion of Proposition 7.7, we conclude that

- $b^{\lambda}_{j,s} = 0$ when j is odd;
- $b^{\lambda}_{2k,s} = (k!)^{-1} s^k b^{\lambda,(k)}_{2k}$.

This simple remark has two important consequences. First of all, it simplifies the algebraic relations (7.9) of Proposition 7.7. Second, since flat measures are uniform and conical, (8.4) holds for any flat measure as well. Recall the definition of the moments $s^k b^{\lambda,(k)}_{2k}$. It is not difficult to see that (8.4) implies the following identity for every conical $\lambda \in \mathcal{U}^m(\mathbb{R}^n)$ and for every m–dimensional linear plane $V \subset \mathbb{R}^n$:

$$\int e^{-s|z|^2} \langle z, x\rangle^j \, d\lambda \;=\; \int e^{-s|z|^2} \langle z, x\rangle^j \, d\mathscr{H}^m \llcorner V\,. \tag{8.5}$$

A standard density argument allows us to generalize (8.5) to point (iii) of the following

Lemma 8.6. *If $\lambda \in \mathcal{U}^m(\mathbb{R}^n)$ is conical, then:*

(i) $b^{\lambda}_{2k-1,s} = 0$ *and* $b^{\lambda}_{2k,s} = (k!)^{-1} s^k b^{\lambda,(k)}_{2k}$ *(therefore $b^{\lambda}_{2k,s}$ has only one nontrivial term in the Taylor expansion).*
(ii) $\mathrm{supp}\,(\lambda) \subset \big\{ b^{(k)}_{2k}(x^{2k}) = |x|^{2k} \big\}$*;*
(iii) *For every $u \in \mathrm{supp}\,(\lambda)$, every $f \in \mathbb{R}^m$ with $|f| = |u|$ and every nonnegative Borel function $\varphi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ we have*

$$\int_{\mathbb{R}^n} \varphi\big(|z|, \langle z, u\rangle\big)\, d\lambda(z) \;=\; \int_{\mathbb{R}^m} \varphi\big(|x|, \langle x, f\rangle\big)\, d\mathscr{L}^m(x)\,. \tag{8.6}$$

We now focus on the algebraic relation (ii) for $k = 1$. In this case we have

$$b^{(1)}_2(x^2) \;=\; |x|^2 \qquad \text{for every } x \in \mathrm{supp}\,(\lambda). \tag{8.7}$$

Consider an orthonormal system of vectors $e_1, \ldots, e_n$ which diagonalizes the symmetric bilinear form $b^{(1)}_2$:

$$b^{(1)}_2(x \odot y) \;=\; \alpha_1 \langle x, e_1\rangle \langle y, e_1\rangle + \ldots + \alpha_n \langle x, e_n\rangle \langle y, e_n\rangle\,. \tag{8.8}$$

The bilinear form is positive semidefinite and we fix the convention that the eigenvalues α_i are ordered as $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \geq 0$. A simple computation using (8.6) implies that $\mathrm{tr}\, b^{(1)}_2 = \mathrm{tr}\, b^{\lambda}_{2,1} = m$:

Lemma 8.7. *Let $\lambda \in \mathcal{U}^m(\mathbb{R}^n)$ be conical. Then $\mathrm{tr}\, b^{(1)}_2 = \mathrm{tr}\, b^{\lambda}_{2,1} = m$.*

Proof. From point (i) of Lemma 8.6 we have $\operatorname{tr} b_2^{(1)} = \operatorname{tr} b^\lambda_{2,1}$. Using point (iii) of Lemma 8.6, we can compute

$$\begin{aligned}\operatorname{tr}(b^\lambda_{2,1}) &= \sum_{i=1}^n b^\lambda_{2,1}(e_i^2) \;=\; 2I(1)^{-1}\int e^{-|z|^2}\sum_{i=1}^n \langle e_i, z\rangle^2\, d\lambda(z)\\ &= 2I(1)^{-1}\int e^{-|z|^2}|z|^2\, d\lambda(z) \;=\; 2I(1)^{-1}\int_{\mathbb{R}^m} e^{-|x|^2}|x|^2\, d\mathscr{L}^m(x)\,.\end{aligned}$$

This last integral can be easily evaluated with an integration by parts (see for instance Proposition B.1) and gives $\operatorname{tr}(b^\lambda_{2,1}) = m$. □

Thanks to this observation, the crucial step in proving Proposition 8.5 is the inequality

$$\alpha_m \geq 1\,. \tag{8.9}$$

Indeed this inequality and Lemma 8.7 yield that $\alpha_1 = \ldots = \alpha_m = 1$ and $\alpha_{m+1} = \ldots = \alpha_n = 0$. Therefore, if we denote by V the vector space spanned by $e_1, \ldots, e_m$, we obtain

$$b_2^{(1)}(x^2) \;=\; |P_V(x)|^2 \qquad \text{for every } x.$$

Coming back to (8.7), we discover that $|P_V(x)|^2 = |x|^2$ for every $x \in \operatorname{supp}(\lambda)$, namely that the support of λ is contained in the m–dimensional plane V. This implies the desired claim $\lambda = \mathscr{H}^m \llcorner V$.

The inequality (8.9) is always satisfied when $m \leq 2$, whereas for $m \geq 3$ it is implied by the additional hypothesis (8.3). The argument of Preiss which elads to (8.9) is elementary and uses again point (iii) of Lemma 8.6.

Plan of the chapter. In section 1 we prove the conical properties of the tangent measure at infinity; in section 2 we prove Lemma 8.6 and in section 3 we prove Proposition 8.5.

1. The tangent measure at infinity is a cone

Proof of Corollary 8.4. As observed in the introduction, $\rho^{-m}\lambda_{0,\rho}$ is a tangent measure at infinity to μ. Since $\operatorname{Tan}_m(\mu, \infty) = \{\lambda\}$, we conclude $\rho^{-m}\lambda_{0,\rho} = \lambda$.

We will now prove (8.2). Let $x \in \operatorname{supp}(\lambda)$ and $\rho > 0$. From point (8.1) we conclude that

$$\rho^m \lambda(B_{r/\rho}(x)) \;=\; \lambda\big(B_r(\rho x)\big) \qquad \text{for every } r > 0.$$

Since $\lambda \in \mathcal{U}^m(\mathbb{R}^n)$ and $x \in \operatorname{supp}(\lambda)$, we conclude

$$\lambda\big(B_r(\rho x)\big) \;=\; \omega_m r^m \qquad \text{for every } r > 0,$$

which implies $\rho x \in \operatorname{supp}(\lambda)$. □

2. Conical uniform measures

Proof of Lemma 8.6. **(i)** By definition we have

$$b^\lambda_{j,s}\big(x^j\big) \;=\; \frac{(2s)^j}{j!} I(s)^{-1}\int e^{-s|z|^2}\langle x, z\rangle^j\, d\lambda(z)\,. \tag{8.10}$$

We make the change of variables $w = s^{1/2}z$ and we use the conical property $\lambda_{0,s} = s^m\lambda$ to conclude

$$b^\lambda_{j,s}\big(x^j\big) \;=\; \frac{(2s)^j}{j!} I(s)^{-1} s^{-j/2-m/2}\int e^{-|w|^2}\langle x, w\rangle^j\, d\lambda(w)\,.$$

From Lemma 7.2 and Proposition B.1, it follows that

$$I(s) \;=\; \left(\frac{\pi}{s}\right)^{m/2}.$$

Therefore, we conclude

$$b^{\lambda}_{s,j}\big(x^j\big) \;=\; \frac{2^j\, s^{j/2}}{\pi^{m/2} j!} \int e^{-|w|^2} \langle x, w\rangle^j \, d\lambda(w)\,. \tag{8.11}$$

From (7.7) of Proposition 7.7 we conclude that:

$$b^{\lambda}_{s,j}(x^j) \;=\; \begin{cases} 0 & \text{if } j \text{ is odd} \\ \\ ((j/2)!)^{-1} s^{j/2}\, b^{\mu,(j/2)}_j(x^j) & \text{if } j \text{ is even.} \end{cases}$$

Since the values of a symmetric j–linear form b are determined by its values on the elements of the form x^j, from these identities we obtain (i).

(ii) From (i) and (7.7) of Proposition 7.7 (applied to λ) we obtain

$$b^{\lambda,(k)}_j \;=\; 0 \qquad \text{if } j \neq 2k. \tag{8.12}$$

From (7.9) we have

$$\sum_{i=1}^{2k} b^{\lambda,(k)}_i(x^i) \;=\; |x|^{2k} \qquad \text{for every } x \in \operatorname{supp}(\lambda). \tag{8.13}$$

Clearly (8.12) and (8.13) give (ii).

(iii) From (i) we know that

$$\int e^{-s|z|^2} \langle z, u\rangle^{2k-1}\, d\lambda(z) \;=\; 0 \qquad \text{for every } u \in \mathbb{R}^n.$$

From (i) we also know

$$b^{\lambda}_{s,2j}(x^{2j}) \;=\; \frac{s^j}{j!}\, b^{\lambda,(j)}_{2j}(x^{2j})\,. \tag{8.14}$$

Thus, we can compute

$$\begin{aligned} & \int e^{-s|z|^2} \langle z, u\rangle^{2k}\, d\lambda(z) \\ \overset{(8.10)}{=} \quad & \left(\frac{\pi}{s}\right)^{m/2} \frac{(2k)!}{2^{2k} s^{2k}} b^{\lambda}_{2k,s}(u^{2k}) \\ \overset{(8.14)}{=} \quad & \left(\frac{\pi}{s}\right)^{m/2} \frac{(2k)!}{k!\, 2^{2k} s^k} b^{\lambda,(k)}_{2k}(u^{2k}) \\ \overset{(8.12)+(8.13)}{=} \quad & \left(\frac{\pi}{s}\right)^{m/2} \frac{(2k)!}{k!\, 2^{2k} s^k}\, |u|^{2k} \qquad \text{for every } u \in \operatorname{supp}(\lambda). \end{aligned} \tag{8.15}$$

Now fix an orthonormal basis $e_1, \ldots, e_m$ on $\mathbb{R}^m$ and consider the vector $f := |u| e_1$. Then we have

$$\int_{\mathbb{R}^m} e^{-s|x|^2} \langle x, f\rangle^j \, d\mathscr{L}^m(x) \;=\; |u|^j \int_{\mathbb{R}^{m-1}} e^{-s|\xi|^2}\, d\mathscr{L}^{m-1}(\xi) \int_{\mathbb{R}} e^{-s|t|^2} t^j \, d\mathscr{L}^1(t)\,. \tag{8.16}$$

This integral is equal to 0 when j is odd. When j is even we can use Proposition B.1 to compute it and we conclude that it is equal to

$$\left(\frac{\pi}{s}\right)^{m/2}\frac{(2k)!}{k!\,2^{2k}s^k}\,|u|^{2k}\,.$$

Therefore, the integrals in (8.15) and (8.16) are equal.

Since this identity is independent of the choice of $e_1,\ldots,e_m$, we conclude that the following equality

$$\int e^{-s|z|^2}\langle z,u\rangle^j\,d\lambda(z) \;=\; \int_{\mathbb{R}^m} e^{-s|x|^2}\langle x,f\rangle^j\,d\mathscr{L}^m(x) \tag{8.17}$$

holds for every $s>0$, every $u\in\operatorname{supp}(\lambda)$, every $f\in\mathbb{R}^m$ with $|u|=|e|$ and every $j\in\mathbb{N}$.

Let $Y\subset\mathbb{R}^2$ be the set $\{y_1\geq 0\}$ and denote by $\mathcal{B}$ the set of Borel functions $\varphi : Y\to\mathbb{R}$ such that $\varphi(|z|,\langle z,u\rangle)\in L^1(\mathbb{R}^n,\lambda)$ and

$$\int \varphi(|z|,\langle z,u\rangle)\,d\lambda(z) \;=\; \int_{\mathbb{R}^m}\varphi(|x|,\langle x,f\rangle)\,d\mathscr{L}^m(x) \tag{8.18}$$

holds for every $u\in\operatorname{supp}(\lambda)$ and every $f\in\mathbb{R}^m$ with $|u|=|f|$. From a standard approximation argument the claim (iii) of the lemma follows from the inclusion

$$C_c(Y)\;\subset\;\mathcal{B}\,. \tag{8.19}$$

To show that (8.19) holds, we use an approximation argument similar to those exploited in the proof of Proposition 6.11 (see also Lemma 6.14) and in the proof of Proposition 7.1.

First of all, from (8.17) we conclude that $\mathcal{B}$ contains all functions of the form $e^{-sy_1^2}y_2^j$. By taking k times the derivative in s of both sides of (8.17) we conclude that $\mathcal{B}$ contains all functions of type

$$e^{-sy_1^2}y_1^{2k}y_2^j \qquad \text{for } s>0 \text{ and } j,k\in\mathbb{N}.$$

Moreover, $\mathcal{B}$ is a vector space. Therefore, for every k, j, and N, $\mathcal{B}$ contains the functions

$$e^{-sy_1^2}y_1^{2k}y_2^j\left(\sum_{i=0}^N\frac{s^iy_2^{2i}}{i!}\right)\,.$$

Using that $\lambda(B_r(0))=\mathscr{L}^m(B_r(0))$, we can apply the Dominated Convergence Theorem to show that

$$\lim_{N\uparrow\infty}\int e^{-s|z|^2}|z|^{2k}\langle f,z\rangle^j\left(\sum_{i=0}^N\frac{s^i\langle f,z\rangle^{2i}}{i!}\right)d\mathscr{L}^m(z) \;=\; \int e^{-s(|z|^2+\langle z,f\rangle^2)}|z|^{2k}\langle f,z\rangle^j\,d\mathscr{L}^m(z)$$

$$\lim_{N\uparrow\infty}\int e^{-s|z|^2}|z|^{2k}\langle u,z\rangle^j\left(\sum_{i=0}^N\frac{s^i\langle u,z\rangle^{2i}}{i!}\right)d\lambda(z) \;=\; \int e^{-s(|z|^2+\langle z,u\rangle^2)}|z|^{2k}\langle u,z\rangle^j\,d\lambda(z)\,.$$

Therefore, we conclude that any linear combination of functions of type $e^{-s|y|^2}y_1^{2k}y_2^j$ with positive s belongs to $\mathcal{B}$.

Now we fix $\varphi\in C_c(Y)$ and we denote by $\mathcal{C}$ be the vector space generated by the functions $f\in C(Y)$ of type

$$a+e^{-s|y|^2}Q(y_1^2,y_2)$$

where Q are polynomials, a real constants, and s positive constants.

We let $X = Y \cup \{\infty\}$ be the one–point compactification of Y, we set $\psi(y_1, y_2) := e^{|y|^2}\varphi(y_1, y_2)$, and we extend it to $\tilde{\psi} \in C(X)$ by $\tilde{\psi}(\infty) = 0$. Any function $f \in \mathcal{C}$ has a unique extension $\tilde{f} \in C(X)$. The set $\tilde{\mathcal{C}}$ of such extensions is an algebra, it separates the points, and it vanishes at no point. Therefore, we can use the Stone–Weierstrass Theorem to find a sequence $\{\tilde{f}_i\} \subset \tilde{\mathcal{C}}$ which converges uniformly to $\tilde{\psi}$. If $\{f_i\}$ is the corresponding sequence of $\mathcal{C}$, we then conclude that

$$g_i(y_1, y_2) \;:=\; e^{-|y|^2} f_i(y_1, y_2)$$

converge uniformly to φ and $g_i(y) \leq Ce^{-|y|^2}$ for some constant C. From that which has been proved above, we have $g_i \in \mathcal{B}$. The bound $g_i(y) \leq Ce^{-|y|^2}$, together with $\lambda(B_r(0)) = \mathscr{L}^m(B_r(0))$, implies that

$$\lim_{i\uparrow\infty} \int g_i(|z|, \langle z, u\rangle)\, d\lambda(z) \;=\; \int \varphi(|z|, \langle z, u\rangle)\, d\lambda(z)$$

$$\lim_{i\uparrow\infty} \int_{\mathbb{R}^m} g_i(|x|, \langle x, f\rangle)\, d\mathscr{L}^m(x) \;=\; \int_{\mathbb{R}^m} \varphi(|x|, \langle x, f\rangle)\, d\mathscr{L}^m(x)\,.$$

Hence, we finally obtain that $\varphi \in \mathcal{B}$. □

3. Proof of Proposition 8.5

Proof of Proposition 8.5. It is trivial to show that $\mathcal{U}^0(\mathbb{R}^n)$ consists of the Dirac mass concentrated at the origin, and therefore in this case the Proposition is trivially true.

Let $m \geq 1$ and consider the bilinear form $b_2^{(1)}$. We select n orthonormal vectors $e_1, \ldots e_n$ such that

$$b_2^{(1)}(x \odot y) \;=\; \alpha_1 \langle x, e_1\rangle\langle y, e_1\rangle + \ldots + \alpha_n \langle x, e_n\rangle\langle y, e_n\rangle\,, \tag{8.20}$$

where we fix the convention that $\alpha_1 \geq \ldots \geq \alpha_n \geq 0$ (recall that $b_2^{(1)} = b_{2,1}^{\lambda}$, which is positive semidefinite by definition). We claim that

$$\alpha_m \geq 1\,. \tag{8.21}$$

From (8.21) and Lemma 8.7 we would conclude $\alpha_1 = \ldots \alpha_m = 1$ and $\alpha_{m+1} = \ldots = \alpha_n = 0$. Therefore, if we denote by V the vector space generated by $e_1, \ldots, e_m$, we would conclude that $b_2^{(1)}(x^2) = |P_V(x)|^2$. Hence, from point (ii) of Lemma 8.6 we would conclude $\mathrm{supp}\,(\lambda) \subset \{|x|^2 = |P_V(x)|^2\}$, that is $\mathrm{supp}\,(\lambda) \subset V$. This would imply $\lambda = \mathscr{H}^m \llcorner V$, which is the desired claim (compare with Remark 3.14). Therefore, it remains to show that (8.21) holds.

Case $m = 1, 2$**.** In this case λ cannot be a Dirac mass and thus $\mathrm{supp}\,(\lambda) \neq \{0\}$. If $x \in \mathrm{supp}\,(\lambda)$, since λ is conical we have $z := x/|x| \in \mathrm{supp}\,(\lambda)$. Therefore, $b_2^{(1)}(z^2) = |z|^2 = 1$. Hence, we have the inequality

$$\alpha_1 \;\geq\; \sup_{|z|=1} b_2^{(1)}(z^2) \;\geq\; 1\,,$$

which proves (8.21) for $m = 1$.

When $m=2$, let z be as above and let f be a vector of $\mathbb{R}^2$ with modulus $1=|z|$. Using Lemma 8.6 we can write

$$\begin{aligned}\lambda\big(\{y:|\langle y,z\rangle|\leq 1\}\big) &= \int \mathbf{1}_{|\langle y,z\rangle|\leq 1}\,d\lambda(y)\\ &= \int_{\mathbb{R}^2}\mathbf{1}_{|\langle x,f\rangle|\leq 1}\,d\mathscr{L}^2(x) \;=\; \infty\,.\end{aligned}$$

Hence, we conclude that there exists a sequence $\{z'_j\}\subset \mathrm{supp}\,(\lambda)$ with

$$\lim_{n\uparrow\infty}|z'_j| \;=\; \infty \qquad\qquad |\langle z'_j,z\rangle| \;\leq\; 1\,.$$

By passing to a subsequence (not relabeled) we can assume that $y_j := z'_j/|z'_j|$ converge to a $y\in\mathbb{R}^n$ with $|y|=1$. Then we would have

$$|\langle y,z\rangle| \;=\; \lim_{n\uparrow\infty}\frac{|\langle z'_j,z\rangle|}{|z'_j|} \;\leq\; 0\,.$$

Since $y_j\in\mathrm{supp}\,(\lambda)$, we know $b^{(1)}_2(y_j^2)=|y_j|^2=1$. Therefore, passing into the limit in j we obtain $b^{(1)}_2(y^2)=1$. Summarizing, we know that

$$|y| \;=\; |z| \;=\; 1\,, \qquad \langle y,z\rangle \;=\; 0 \qquad \text{and} \qquad b^{(1)}_2(z^2) \;=\; b^{(1)}_2(y^2) \;=\; 1\,.$$

This implies $\alpha_2\geq 1$, and hence gives (8.21).

Case $m\geq 3$**.** Let W be any m–dimensional linear plane and assume that $f_1,\ldots f_k$ is an orthonormal base for the orthogonal space $W^\perp$. Then

$$\begin{aligned}\mathrm{tr}\,(b^{(1)}_2\llcorner W^\perp) &= \mathrm{tr}\,(b_{2,1}\llcorner W^\perp) \;=\; \sum_{i=1}^k b_{2,1}(f_i^2)\\ &= 2I(1)^{-1}\int e^{-|z|^2}\sum_{i=1}^k\langle z,f_i\rangle^2\,d\lambda(z)\\ &= C(m)\int e^{-|z|^2}\,\mathrm{dist}^2(z,W)\,d\lambda(z)\,, \end{aligned}\tag{8.22}$$

where $C(m)$ is a constant which depends only on m.

Now, let V be the m–dimensional plane spanned by $e_1,\ldots,e_m$. Then we have

$$\mathrm{tr}\,(b^{(1)}_2\llcorner V^\perp) \;\leq\; \min_{m\text{–planes } W}\,\mathrm{tr}\,(b^{(1)}_2\llcorner W^\perp)$$

because $V^\perp$ is spanned by the $n-m$ eigenvectors of $b^{(1)}_2$ corresponding to the $n-m$ smallest eigenvalues. Thus, using (8.22), we conclude

$$\int e^{-|z|^2}\,\mathrm{dist}^2\,(z,V)\,d\lambda(z) \;=\; \min_{m\text{–planes } W}\int e^{-|z|^2}\,\mathrm{dist}^2\,(z,W)\,d\lambda(z) \;\leq\; \varepsilon\,. \tag{8.23}$$

Now let $\delta>0$: We claim that if ε is sufficiently small, then

$$\text{for all } e\in V\cap B_1(0) \text{ there is } z\in\mathrm{supp}\,(\lambda) \text{ such that } |z-e|\leq\delta. \tag{8.24}$$

First we show that for δ sufficiently small (8.24) yields the statement of the Proposition. Indeed apply (8.24) to $e = e_m$ and let $x \in \mathrm{supp}\,(\lambda)$ be such that $|x - e_m| \leq \delta$. Since $\alpha_i + (m-1)\alpha_m \leq \mathrm{tr}\,(b_2^{(1)}) = m$ for every $i \leq m-1$ (recall Lemma 8.7), we conclude

$$\alpha_i - 1 \;\leq\; (m-1)(1-\alpha_m) \qquad \text{for every } i \leq m-1. \tag{8.25}$$

Moreover, from the definition of α_i we have

$$\alpha_i \;\leq\; \alpha_m \;\leq\; 1 \qquad \text{for every } i \geq m. \tag{8.26}$$

Since $x \in \mathrm{supp}\,(\lambda)$ we have

$$\sum_{i=1}^n \alpha_i \langle x, e_i\rangle^2 \;=\; b_2^{(1)}(x^2) \;=\; |x|^2\,.$$

Therefore,

$$\begin{aligned}
0 \;&=\; \sum_{i=1}^n (\alpha_i - 1)\langle x, e_i\rangle^2 \overset{(8.26)}{\leq} \sum_{i=1}^m (\alpha_i - 1)\langle x, e_i\rangle^2 \\
&\overset{(8.25)}{\leq}\; (m-1)(1-\alpha_m)\sum_{i=1}^{m-1}\langle x, e_i\rangle^2 + (\alpha_m - 1)\langle x, e_m\rangle^2 \\
&=\; (m-1)(1-\alpha_m)\sum_{i=1}^{m-1}\langle x - e_m, e_i\rangle^2 \\
&\qquad -(1-\alpha_m)\big(\langle e_m, e_m\rangle + \langle e_m, x - e_m\rangle\big)^2 \\
&\leq\; (1-\alpha_m)\left((m-1)\sum_{i=1}^{m-1}|x - e_m|^2 - \big(1 - |x - e_m|\big)^2\right) \\
&\leq\; (1-\alpha_m)\big((m-1)^2\delta^2 - (1-\delta)^2\big)\,.
\end{aligned}$$

When δ is sufficiently small, the number $(m-1)^2\delta^2 - (1-\delta)^2$ is negative. Therefore, for δ sufficiently small the inequality above is satisfied if and only if $\alpha_m \geq 1$, which is the desired conclusion.

It remains to prove (8.24). We argue again by contradiction. If the claim were wrong, then we would have a number $\delta > 0$, a sequence of m–uniform measures $\{\lambda_k\}$, a sequence of m–planes $\{V_k\}$, and a sequence of points $\{x_k\}$ such that

$$\lim_{n\uparrow\infty}\int e^{-|z|^2}\,\mathrm{dist}^2\,(z, V_k)\,d\lambda_k(z) \;=\; 0$$

$$x_k \in V_k \qquad \text{and} \qquad \lambda(B_\delta(x_k)) \;=\; 0\,.$$

Since everything is invariant under rotations, we can assume that $V_k = W$ and $x_k = x$. Moreover, we can also assume that $\lambda_k \overset{*}{\rightharpoonup} \lambda$. Then it follows that $\lambda \in \mathcal{U}^m(\mathbb{R}^n)$, $\mathrm{supp}\,(\lambda) \subset V$ and $\lambda(B_\delta(x)) = 0$. On the other hand, the first two conditions imply easily that $\lambda = \mathscr{H}^m \llcorner V$ (compare with Remark 3.14) and hence they are incompatible with the third condition. $\square$

CHAPTER 9

Flatness at infinity implies flatness

In this chapter we will make the last step towards the proof of Preiss' Theorem and we will prove the following proposition:

PROPOSITION 9.1. *Let $\mu \in \mathcal{U}^m(\mathbb{R}^n)$ and V an m–dimensional plane. If $\mathscr{H}^m \llcorner V$ is the tangent measure to μ at infinity, then $\mu = \mathscr{H}^m \llcorner V$.*

The first information which we gain on an uniform measure which is flat at infinity concerns the special moments $b^{(k)}_{2k} = b^{\mu,(k)}_{2k}$.

Recall that from Lemma 7.2 and Proposition B.1 we have

$$I(s) \;=\; \int e^{-s|z|^2}\, d\mathscr{L}^m(z) \;=\; \frac{1}{s^{m/2}} \int e^{-|z|^2}\, d\mathscr{L}^m(z) \;=\; \left(\frac{\pi}{s}\right)^{m/2} .$$

Using this identity and changing variables in the integral defining the moments, one can readily check that

$$\begin{aligned}
\frac{b^{\mu}_{2k,s}(u_1,\ldots,u_{2k})}{s^k} &= r^{2k} b^{\mu}_{2k,r^{-1/2}}(u_1,\ldots,u_{2k}) \\
&= \frac{2^{2k}}{(2k)!} I(1)^{-1} \int \langle w,u_1\rangle \ldots \langle w,u_{2k}\rangle\, e^{-|w|^2} d\left[\frac{\mu_{0,r}}{r^m}\right](w)\,.
\end{aligned}$$

Since $\mathscr{H}^m \llcorner V$ is tangent to μ at infinity, by letting $s \downarrow 0$ (and hence $r \uparrow \infty$) we gain the identity

$$\begin{aligned}
\lim_{s\downarrow 0} \frac{b^{\mu}_{2k,s}(u_1,\ldots,u_k)}{s^k} &= \frac{2^{2k}}{(2k)!} I(1)^{-1} \int \langle w,u_1\rangle \ldots \langle w,u_{2k}\rangle\, e^{-|w|^2} d\left[\mathscr{H}^m \llcorner V\right](w) \\
&= b^{\mathscr{H}^m \llcorner V}_{2k,1}(u_1,\ldots,u_{2k})\,.
\end{aligned}$$

Since Proposition 7.7 gives

$$\lim_{s\downarrow 0} \frac{b^{\mu}_{2k,s}(u_1,\ldots,u_{2k})}{s^k} \;=\; \frac{b^{\mu,(k)}_{2k}(u_1,\ldots,u_{2k})}{k!}\,,$$

we can compute

$$b^{(k)}_{2k}(x^{2k}) \;=\; k! b^{\mathscr{H}^m \llcorner V}_{2k,1}(x^{2k}) \;=\; \frac{2^{2k} k! |P_V(x)|^{2k}}{(2k)! I(1)} \int_{\mathbb{R}^{m-1}} e^{-|\xi|^2}\, d\mathscr{L}^{m-1}(\xi) \int_{\mathbb{R}} e^{-t^2} t^{2k}\, d\mathscr{L}^1(t)\,.$$

Here and in what follows, we use the notation of Definition 4.5: P_V denotes the orthogonal projection on V and Q_V the orthogonal projection on $V^\perp$. The integral above can be computed explicitly (see for instance Proposition B.1) and we obtain point (i) of Lemma 9.2 below. The identity $k! b^{\mathscr{H}^m \llcorner V}_{2k,1} = b^{\mathscr{H}^m \llcorner V,(k)}_{2k}$ can be proved with a direct computation or one can use Lemma 8.6.

Point (ii) of Lemma 9.2 provides information on the moments $b^{(k)}_{2k-1}$. Its proof is less direct but not long. Here, whenever b is a symmetric j–linear form, we denote by $b \llcorner V$ the restriction of b to $\bigodot^j V$.

LEMMA 9.2. *Let μ and V be as in Proposition 9.1. Then*

(i) $b^{(k)}_{2k}(x^{2k}) = k! b^{\mathscr{H}^m \llcorner V}_{2,1}(x^{2k}) = b^{\mathscr{H}^m \llcorner V,(k)}_{2k}(x^{2k}) = |P_V(x)|^{2k}$ *for every* $x \in \mathbb{R}^n$.

(ii) $b^{(k)}_{2k-1} \llcorner V = 0$ *for every* k.

Note that the case $k = 1$ of (ii) implies the existence of a vector $w \in V^\perp$ such that $b^{(1)}_1(v) = \langle v, w\rangle$ for every $v \in \mathbb{R}^n$. In order to simplify some computations it is useful to introduce the following notation

DEFINITION 9.3. *We let $b \in V^\perp$ be such that $b^{(1)}_1(z) = 2\langle b, z\rangle$.*

Hausdorff distance between $\operatorname{supp}(\mu)$ **and** V**.** Recall that the moments $b^{(k)}_i$ satisfy the following identities:

$$\sum_{i=1}^{2k} b^{(k)}_i(x^i) \;=\; |x|^{2k} \qquad \text{for every } x \in \operatorname{supp}(\mu). \tag{9.1}$$

By Lemma 9.2(i), the case $k = 1$ of (9.1) gives

$$2\langle b, x\rangle + |P_V(x)|^2 \;=\; |x|^2 \qquad \text{for every } x \in \operatorname{supp}(\mu),$$

which becomes

$$|Q_V(x)|^2 \;=\; 2\langle b, x\rangle \qquad \text{for every } x \in \operatorname{supp}(\mu). \tag{9.2}$$

Combining Lemma 9.2(ii) and (9.2) (recall that $b \in V^\perp$), we obtain

$$2|b||Q_V(x)| \;\geq\; 2\langle b, Q_V(x)\rangle \;=\; 2\langle b, x\rangle \;=\; |Q_V(x)|^2$$

for every $x \in \operatorname{supp}(\mu)$. Therefore, we conclude that $|Q_V(x)| \;\leq\; 2|b|$ for every $x \in \operatorname{supp}(\mu)$. That is, the distance between any point $x \in \operatorname{supp}(\mu)$ and the plane V is uniformly bounded by a constant. It is not difficult to show that also the distance between $\operatorname{supp}(\mu)$ and any $v \in V$ is bounded. These two conclusions are incorporated in the following lemma.

LEMMA 9.4. *Let μ and V be as in Proposition 9.1. Then*

(iii) $b^{(1)}_1(x) = |Q_V(x)|^2$ *and* $|Q_V(x)| \leq \|b^{(1)}_1\|$ *for every* $x \in \operatorname{supp}(\mu)$.

(iv) *There exists* $r_0 > 0$ *such that* $\operatorname{dist}(v, \operatorname{supp}(\mu)) < r_0$ *for every* $v \in V$.

Coming back to (9.2), note that if we could prove $b = 0$, then we would conclude $\operatorname{supp}(\mu) \subset V$. As already observed many times, this would imply $\mu = \mathscr{H}^m \llcorner V$; compare with Remark 3.14. (On the other hand, if $\mu = \mathscr{H}^m \llcorner V$, then

$$b^{\mu}_{1,s}(u) \;=\; b^{\mathscr{H}^m \llcorner V}_{1,s}(u) \;=\; 2sI(s)^{-1} \int_V \langle z, u\rangle \, d\mathscr{H}^m(z) \;=\; 0\,,$$

and hence necessarily $b^{(1)}_1 = 0$). Therefore, our goal is to show that $b = 0$ (or, equivalently, $b^{(1)}_1 = 0$). In order to prove this we will make use of the case $k = 2$ in (9.1) and many computations.

Properties of $\operatorname{tr} b^{(2)}_2$**.** The basic observations are contained in the following proposition.

PROPOSITION 9.5. *Let μ be as in Proposition 9.1. Then*

$$\operatorname{tr} b_2^{(2)} = 0 \tag{9.3}$$

$$\operatorname{tr} b_2^{(2)} \geq \frac{4}{m+2}\left|b_1^{(1)}\right|^2 . \tag{9.4}$$

The proof of (9.3) is fairly simple.

PROOF OF (9.3). From Proposition 7.7 we have

$$b_{2,s} = sb_2^{(1)} + \frac{s^2}{2}b_2^{(2)} + o(s^2)\,.$$

Therefore

$$\operatorname{tr} b_2^{(2)} = 2\lim_{s\downarrow 0}\frac{\operatorname{tr}\,(s^{-1}b_{2,s}) - \operatorname{tr} b_2^{(1)}}{s}\,. \tag{9.5}$$

From Lemma 9.2(i) we conclude that $b_2^{(1)}(u,v) = \langle P_V(u), P_V(v)\rangle$ (indeed if we define the bilinear form $B(u,v) := \langle P(u), P(v)\rangle$, then Lemma 9.2(i) says that the quadratic forms induced by $b_2^{(1)}$ and B are the same). Thus, $\operatorname{tr} b_2^{(1)}$ is m, i.e. the dimension of the linear space V. Recall the definition of $b_{2,s}$. If we fix an orthonormal system of vectors $e_1,\ldots,e_n$ on $\mathbb{R}^n$, then we have

$$\begin{aligned}
\operatorname{tr}\,\left(s^{-1}b_{2,s}\right) &= s^{-1}\sum_{i=1}^n b_{2,s}(e_i^2) = s^{-1}\frac{(2s)^2}{2!}I(s)^{-1}\int\sum_{i=1}^n e^{-s|z|^2}\langle z, e_i\rangle^2\,d\mu \\
&= 2s\frac{\int |z|^2 e^{-s|z|^2}\,d\mu(z)}{\int e^{-s|z|^2}\,d\mu(z)} = 2s\frac{\int_{\mathbb{R}^m}|x|^2 e^{-s|x|^2}\,d\mathscr{L}^m(x)}{\int_{\mathbb{R}^m} e^{-s|x|^2}\,d\mathscr{L}^m(x)}\,,
\end{aligned}$$

where the last equality follows from Lemma 7.2. Using Proposition B.1, we obtain

$$\operatorname{tr}\,\left(s^{-1}b_{2,s}\right) = m\,,$$

and therefore, plugging this into (9.5) we conclude (9.3). □

The inequality (9.4) will be split into two parts:

$$\operatorname{tr}\,\left(b_2^{(2)}\llcorner V^\perp\right) = 2\left|b_1^{(1)}\right|^2 \tag{9.6}$$

$$\operatorname{tr}\,\left(b_2^{(2)}\llcorner V\right) \geq -\frac{2m}{m+2}\left|b_1^{(1)}\right|^2 . \tag{9.7}$$

The first part is not complicated to prove.

PROOF OF (9.6). We use again

$$b_{2,s} = sb_2^{(1)} + \frac{s^2}{2}b_2^{(2)} + o(s^2)$$

to conclude

$$\begin{aligned}
\operatorname{tr}\,\left(b_2^{(2)}\llcorner V^\perp\right) &= 2\lim_{s\downarrow 0}\frac{\operatorname{tr}\,\left(b_{2,s}\llcorner V^\perp\right) - s\operatorname{tr}\,\left(b_2^{(1)}\llcorner V^\perp\right)}{s^2} \\
&= 2\lim_{s\downarrow 0}\frac{\operatorname{tr}\,\left(b_{2,s}\llcorner V^\perp\right)}{s^2}\,,
\end{aligned} \tag{9.8}$$

where, in the last equality, we have used Lemma 9.2(i).

Let $e_1, \dots, e_{n-m}$ be a system of orthonormal vectors of $V^\perp$. The same calculations performed in the proof of (9.3) yield

$$\begin{aligned}\operatorname{tr}\left(b_{2,s}\llcorner V^\perp\right) &= 2s^2 I(s)^{-1}\int e^{-s|z|^2}\sum_{i=1}^{n-m}\langle z,e_i\rangle^2\,d\mu\\ &= 2s^2 I(s)^{-1}\int e^{-s|z|^2}|Q_V(z)|^2\,d\mu\,. \end{aligned}\tag{9.9}$$

From Lemma 9.4(iii) we know that

$$|Q_V(z)|^2 = b_1^{(1)}(z) \qquad \text{for every } z\in \operatorname{supp}(\mu)\,. \tag{9.10}$$

Recalling that

$$b_{1,s}(z) = s b_1^{(1)}(z) + o(s) = 2s\langle b,z\rangle + o(s)\,, \tag{9.11}$$

we can write

$$\begin{aligned}\operatorname{tr}\left(b_2^{(2)}\llcorner V^\perp\right) &\overset{(9.8)}{=} 2\lim_{s\downarrow 0}\frac{\operatorname{tr}\left(b_{2.s}\llcorner V^\perp\right)}{s^2} \overset{(9.9)\&(9.10)}{=} 4\lim_{s\downarrow 0} I(s)^{-1}\int e^{-s|z|^2} b_1^{(1)}(z)\,d\mu(z)\\ &= 8\lim_{s\downarrow 0} I(s)^{-1}\int e^{-s|z|^2}\langle b,z\rangle\,d\mu(z)\\ &= 4\lim_{s\downarrow 0}\frac{b_{1,s}(b)}{s} \overset{(9.11)}{=} 8|b|^2 = 2\left|b_1^{(1)}\right|^2\,. \end{aligned}\tag{9.12}$$

□

The last inequality (9.7) is the hard core of the proof of Proposition 9.1. After proving in Section 1 statement (ii) of Lemma 9.2 and statement (iv) of Lemma 9.4, in Section 2 we will introduce some notation and derive an integral formula for $\operatorname{tr}\left(b_2^{(2)}\llcorner V\right)$ (see equation (9.14) of Lemma 9.9). In Section 3 we will study the identity (9.1) when $k=2$ and prove an intermediate inequality involving the integrand of (9.14) (see Lemma 9.10). Finally, in Section 4 we will use these two ingredients in order to prove (9.7).

1. Proofs of (ii) and (iv)

PROOF OF (II). Since $b_{2k-1}^{(k)}$ is symmetric, it suffices to show

$$b_{2k-1}^{(k)}(y^{2k-1}) = 0 \text{ for every } y\in V.$$

Therefore, let us fix $y\in V$ with $y\neq 0$. Since

$$r^{-m}\mu_{0,r} \overset{*}{\rightharpoonup} \mathscr{H}^m\llcorner V \qquad \text{for } r\uparrow\infty,$$

there exists a sequence $\{x_j\}\subset \operatorname{supp}(\mu)$ such that

$$\frac{x_j}{|x_j|}\to\frac{y}{|y|} \qquad \text{and} \qquad |x_j|\to\infty\,.$$

Recall that from Proposition 7.7 we have

$$b_{2k-1}^{(k)}(x_j^{2k-1}) = |x_j|^{2k} - b_{2k}^{(k)}(x_j^{2k}) - \sum_{i=1}^{2k-2} b_i^{(k)}(x_j^i)$$

(here we adopt the convention that the last sum is equal to 0 if $k=1$). Hence, from Lemma 9.2(i) we obtain

$$b^{(k)}_{2k-1}(x_j^{2k-1}) \;=\; |x_j|^{2k} - |P_V(x_j)|^{2k} - \sum_{i=1}^{2k-2} b^{(k)}_i(x^i_j) \;\geq\; -\sum_{i=1}^{2k-2} b^{(k)}_i(x^i_j)\,.$$

Thus,

$$\begin{aligned} b^{(k)}_{2k-1}(y^{2k-1}) &= |y|^{2k-1}\lim_{j\uparrow\infty}|x_j|^{-(2k-1)}b^{(k)}_{2k-1}(x_j^{2k-1}) \\ &\geq -|y|^{2k-1}\lim_{j\uparrow\infty}|x_j|^{-(2k-1)}\sum_{i=1}^{2k-2} b^{(k)}_i(x^i_j)\,. \end{aligned}$$

Clearly, there exist constants C' and C such that $b^{(k)}_i(x^i_j) \leq C'|x^i_j|^i \leq C(1+|x^i_j|^{2k-2})$ for every $i\in\{1,\ldots,2k-2\}$. Therefore we conclude

$$b^{(k)}_{2k-1}(y^{2k-1}) \;\geq\; -C|y|^{2k-1}\lim_{j\uparrow\infty}\frac{1+|x_j|^{2k-2}}{|x_j|^{2k-1}} \;=\; 0\,.$$

Since $-y\in V$, the same argument gives

$$-b^{(k)}_{2k-1}(y^{2k-1}) \;=\; b^{(k)}_{2k-1}\big((-y)^{2k-1}\big) \;\geq\; 0\,,$$

and we conclude that $b^{(k)}_{2k-1}(y^{2k-1})=0$. □

Proof of (iv). Assume by contradiction that the statement is false. Then there exists $\{x_k\}\subset V$ with

$$r_k \;:=\; \mathrm{dist}\,(\mathrm{supp}\,(\mu), x_k) \;\to\; \infty\,.$$

Let $y_k \in \mathrm{supp}\,(\mu)$ be such that $|x_k-y_k| = r_k$. From Lemma 9.4(iii) it follows that $\mathrm{dist}\,(y_k,V) \leq \|b^{(1)}_1\|$. Let $z_k\in V$ be such that $|y_k-z_k| = \mathrm{dist}\,(y_k,V)$ and consider the measures

$$\mu^k \;:=\; r_k^{-m}\mu_{z_k,r_k}\,.$$

After possibly extracting a subsequence, we can assume that $\mu^k \overset{*}{\rightharpoonup} \mu^\infty$. Note that μ^k satisfies the condition

$$\mu^k(B_r(x)) \;=\; \omega_m r^m \qquad \text{for every } x\in\mathrm{supp}\,(\mu^k) \text{ and } r>0. \tag{9.13}$$

It might be that $\mu^k\notin\mathcal{U}^m(\mathbb{R}^n)$, since we do not know whether the origin belongs to $\mathrm{supp}\,(\mu^k)$. However we know that

$$\mathrm{dist}\,(0,\mathrm{supp}\,(\mu^k)) \;=\; r_k^{-1}\mathrm{dist}\,(z_k,\mathrm{supp}\,(\mu)) \;=\; r_k^{-1}\mathrm{dist}\,(z_k,y_k) \;\leq\; r_k^{-1}|b^{(1)}_1|\,.$$

This, combined with (9.13), gives $0\in\mathrm{supp}\,(\mu^\infty)$ and therefore $\mu^\infty\in\mathcal{U}^m(\mathbb{R}^n)$. On the other hand

$$\mathrm{supp}\,(\mu^k) \;\subset\; \big\{|Q(x)| \;\leq\; |b^{(1)}_1|/r_k\big\}$$

and hence we conclude $\mathrm{supp}\,(\mu^\infty)\subset V$. This, together with $\mu^\infty\in\mathcal{U}^m(\mathbb{R}^n)$, implies $\mu^\infty = \mathscr{H}^m\llcorner V$ (compare with Remark 3.14).

If we set $w_k := x_k - z_k$, we have that

$$\lim_{n\uparrow\infty}\frac{|w_k|}{r_k} \;=\; 1\,.$$

Therefore, we can assume that w_k/r_k converges to a $u \in V$. Since $\mu(B_{r_k}(x_k)) = 0$, we obtain $\mu^k(B_1(w_k/r_k)) = 0$ and therefore we conclude $\mu^\infty(B_1(u)) = 0$, which contradicts $\mu^\infty = \mathscr{H}^m \llcorner V$. □

2. An integral formula for $\mathrm{tr}\left(b_2^{(2)} \llcorner V\right)$

We start this section by introducing some notation.

Definition 9.6. *We let γ be the measure $(2\pi)^{-m/2} e^{-|z|^2/2} \mathscr{H}^m \llcorner V$.*

Next we consider two linear maps $\omega : \bigodot^2 V \to \mathbb{R}^n$ and $\hat{b} \in \mathrm{Hom}\left(\bigodot^2 V, \mathbb{R}\right)$, as defined below.

Definition 9.7. *We let $\omega : \bigodot^2 V \to \mathbb{R}^n$ be defined by*

$$\langle \omega(u^2), w\rangle \;:=\; 3b_3^{(2)}(u^2 \odot w) - 4|u|^2\langle b, w\rangle \qquad \textit{for every } u \in V \textit{ and every } w \in \mathbb{R}^n.$$

We let $\hat{b} \in \mathrm{Hom}\left(\bigodot^2 V, \mathbb{R}\right)$ be defined by

$$\hat{b}(u^2) \;:=\; b_2^{(2)}(u^2) + \langle \omega(u^2), b\rangle\,.$$

Remark 9.8. *Note that from Lemma 9.2(ii) and Definition 9.3 it follows that*

$$\langle \omega(u^2), w\rangle \;=\; 0 \qquad \textit{for every } u, w \in V.$$

Hence, $\omega(u^2)$ takes values in $V^\perp$ and we can regard it as a linear map $\omega : \bigodot^2 V \to V^\perp$.

We are now ready to state the formula which is the main goal of this section.

Lemma 9.9. *Let μ and V be as in Proposition 9.1. Then we have*

$$\mathrm{tr}\left(b_2^{(2)} \llcorner V\right) \;=\; \int \hat{b}(v^2)\, d\gamma(v)\,. \tag{9.14}$$

Proof. First of all, recall that $b_2^{(2)} \llcorner V$ is symmetric. Therefore, there exists a system of orthonormal coordinates $v_1, \ldots, v_m \in V$ such that the corresponding orthonormal vectors $e_1, \ldots, e_m$ are eigenvectors of $b_2^{(2)}$ with corresponding real eigenvalues $\beta_1, \ldots, \beta_m$. This means that

$$\begin{aligned}\int b_2^{(2)}(v^2)\, d\gamma(v) \;&=\; \int \left(\beta_1 v_1^2 + \ldots + \beta_m v_m^2\right) d\gamma(v)\\ &=\; \beta_1 + \ldots + \beta_m \;=\; \mathrm{tr}\left(b_2^{(2)} \llcorner V\right)\,,\end{aligned} \tag{9.15}$$

where we have used Proposition B.1 to compute the second integral.

Hence, in order to conclude (9.14), we have to prove that

$$\int \langle \omega(v^2), b\rangle\, d\gamma(v) \;=\; 0\,. \tag{9.16}$$

Step 1 Using the same argument which gives (9.15), we conclude that

$$|P_V(z)|^2 \;=\; \int_V \langle z, v\rangle^2\, d\gamma(v) \qquad \text{for every } z \in \mathbb{R}^n. \tag{9.17}$$

Next, let $v \in V$ and $w \in V^\perp$. We use the definition of the moments $b_{k,s}$ and the expansion (7.7) in Proposition 7.7 to compute

$$\lim_{s\downarrow 0} \frac{8s}{I(s)} \int e^{-s|z|^2} \langle z, v\rangle^2 \langle z-b, w\rangle \, d\mu(z) \;=\; 3b_3^{(2)}(v^2 \odot w) - 4b_2^{(1)}(v^2)\langle b, w\rangle \,.$$

Recalling that $b_2^{(1)}(v^2) = |P_V(v)|^2 = |v|^2$, we then obtain

$$\lim_{s\downarrow 0} \frac{8s}{I(s)} \int e^{-s|z|^2} \langle z, v\rangle^2 \langle z-b, w\rangle \, d\mu(z) \;=\; \langle \omega(v^2), w\rangle \,. \tag{9.18}$$

Therefore, we can compute

$$\begin{aligned}
\int_V \langle \omega(v^2), b\rangle \, d\gamma(v) \;&\overset{(9.18)}{=}\; \int_V \lim_{s\downarrow 0} \frac{8s}{I(s)} \int_V e^{-s|z|^2} \langle z, v\rangle^2 \langle z-b, b\rangle \, d\mu(z) \, d\gamma(v) \\
&=\; \lim_{s\downarrow 0} \frac{8s}{I(s)} \int e^{-s|z|^2} \langle z-b, b\rangle \int_V \langle z, v\rangle^2 \, d\gamma(v) \, d\mu(z) \\
&\overset{(9.17)}{=}\; \lim_{s\downarrow 0} \frac{8s}{I(s)} \int e^{-s|z|^2} |P_V(z)|^2 \langle z-b, b\rangle d\mu(z) \,.
\end{aligned} \tag{9.19}$$

Step 2 Next, consider any $w \in \mathbb{R}^n$. Using again Lemma 9.4(iii) and Lemma 7.2 we obtain

$$\begin{aligned}
\left| \int e^{-s|z|^2} |Q_V(z)|^2 \langle z-b, w\rangle \, d\mu(z) \right| \;&\leq\; 4|b|^2|w| \int e^{-s|z|^2} \big(|z| + |b|\big) \, d\mu(z) \\
&=\; 4|b|^2|w| \int_{\mathbb{R}^m} e^{-s|x|^2} \big(|x| + |b|\big) \, d\mathscr{L}^m(x) \,.
\end{aligned}$$

Therefore, using Proposition B.1 we conclude

$$\lim_{s\downarrow 0} \frac{s}{I(s)} \int e^{-s|z|^2} |Q_V(z)|^2 \langle z-b, w\rangle \, d\mu(z) \;=\; 0 \,. \tag{9.20}$$

Next, we will show that the following limit exists and is 0:

$$\lim_{s\downarrow 0} \frac{s}{I(s)} \int e^{-s|z|^2} |z|^2 \langle z-b, w\rangle \, d\mu(z) \,. \tag{9.21}$$

First of all, if we fix a system of orthonormal vectors $e_1, \ldots e_n$ we can write

$$\lim_{s\downarrow 0} \frac{s}{I(s)} \int e^{-s|z|^2} |z|^2 \langle z-b, w\rangle \, d\mu(z) \;=\; \sum_{i=1}^n \lim_{s\downarrow 0} \frac{s}{I(s)} \int e^{-s|z|^2} \langle z, e_i\rangle^2 \langle z-b, w\rangle \, d\mu(z) \,. \tag{9.22}$$

Using Proposition 7.7, one can easily check that the limit (9.22) can be expressed as a linear combination of $b_3^{(2)}(e_i^2 \odot w)$ and $b_2^{(2)}(e_i^2)$ (by this we mean that the coefficients of this linear combination are independent of the uniform measure μ). Therefore, the limit (9.21) exists.

Next, we write

$$\frac{s}{I(s)} \int e^{-s|z|^2} |z|^2 \langle z-b, w\rangle \, d\mu(z) \;=\; \pi^{-m/2} \frac{\int e^{-s|z|^2} |z|^2 \langle z-b, w\rangle \, d\mu(z)}{s^{-1-m/2}} \;=:\; \pi^{-m/2} \frac{J(s)}{s^{-1-m/2}} \,.$$

Note that

$$\frac{J(s)}{s^{-1-m/2}} \;=\; \frac{-\frac{d}{ds} \int e^{-s|z|^2} \langle z-b, w\rangle \, d\mu(z)}{-\frac{2}{m}\frac{d}{ds} s^{-m/2}} \,.$$

If there exists a vanishing sequence of positive numbers $\{s_k\}$ such that $\{J(s_k)\}$ is bounded, we conclude that $\lim_{s_k} J(s_k)/s_k^{-1-m/2} = 0$, and hence that the limit (9.21) is zero.

If instead $\lim_s J(s) = \infty$, then, recalling that $I(s) = \pi^{m/2} s^{-m/2}$, we can use De L'Hôpital's rule to conclude

$$\begin{aligned}\lim_{s\downarrow 0} \frac{s}{I(s)} \int e^{-s|z|^2} |z|^2 \langle z-b, w\rangle \, d\mu(z) &= \frac{m}{2\pi^{m/2}} \lim_{s\downarrow 0} \frac{\int e^{-s|z|^2} \langle z-b, w\rangle \, d\mu(z)}{s^{-m/2}} \\ &= \frac{m}{2} \lim_{s\downarrow 0} \frac{\int e^{-s|z|^2} \langle z-b, w\rangle \, d\mu(z)}{I(s)} .\end{aligned}$$

Note that

$$\lim_{s\downarrow 0} \frac{1}{I(s)} \int e^{-s|z|^2} \langle z, w\rangle \, d\mu(z) = \frac{1}{2} b_1^{(1)}(w) = \langle b, w\rangle$$

and

$$\frac{1}{I(s)} \int e^{-s|z|^2} \langle b, w\rangle \, d\mu(z) = \langle b, w\rangle .$$

Therefore, we find

$$\lim_{s\downarrow 0} \frac{s}{I(s)} \int e^{-s|z|^2} |z|^2 \langle z-b, w\rangle \, d\mu(z) = 0 .$$

Combining this with (9.20) we obtain

$$\lim_{s\downarrow 0} \frac{s}{I(s)} \int e^{-s|z|^2} |P_V(z)|^2 \langle z-b, w\rangle \, d\mu(z) = 0 . \tag{9.23}$$

In the particular case $w = b$, the last equality can be combined with (9.19) to give (9.16), which completes the proof. □

3. An intermediate inequality

In this section, we use the identity (9.1) when $k = 2$ to derive an inequality involving $\hat{b}$.

LEMMA 9.10. *Let μ and V be as in Proposition 9.1. Then*

(v) $b_1^{(2)}(z) + b_2^{(2)}(z^2) + 3b_3^{(2)}\big((P_V(z))^2 \odot Q_V(z)\big) = |Q_V(z)|^2\big(|Q_V(z)|^2 + 2|P_V(z)|^2\big)$.

(vi) *For every $v \in V$ we have*

$$\Big(\hat{b}(v^2)\Big)^2 \le \left|\omega(v^2)\right|^2 |b|^2 . \tag{9.24}$$

PROOF. **(v)** First of all we prove

$$b_3^{(2)}(v \odot w^2) = b_3^{(2)}(v^3) = b_3^{(2)}(w^3) = 0 \qquad \text{for all } v \in V \text{ and } w \in V^\perp. \tag{9.25}$$

Recall that Lemma 9.2(ii) gives $b_3^{(2)}(v^3) = 0$ for every $v \in V$. Next, let $v \in V$ and $w \in V^\perp$ be given. From Proposition 7.7 we have

$$b_3^{(2)}(v \odot w^2) = \lim_{s\downarrow 0} 2s^{-2} b_{3,s}(v \odot w^2) = \lim_{s\downarrow 0} \frac{8s}{I(s)} \int e^{-s|z|^2} \langle z, v\rangle \langle z, w\rangle^2 \, d\mu(z) . \tag{9.26}$$

Clearly $|\langle z, v\rangle| \leq |z||v|$. Moreover, since $w \in V^\perp$, $\langle z, w\rangle^2 = \langle Q_V(z), w\rangle^2 \leq |Q_V(z)|^2|w|^2$. Recalling Lemma 9.4(iii), we can bound the integrand in (9.26) with $|v||b_1^{(1)}|^2|w|^2$, and thus we obtain

$$\begin{aligned}
|b_3^{(2)}(v \odot w^2)| &\leq \lim_{s\downarrow 0} \frac{8s|v||b_1^{(1)}|^2|w|^2}{I(s)} \int e^{-s|z|^2}|z|\, d\mu(z) \\
&= \lim_{s\downarrow 0} \frac{8s|v||b_1^{(1)}|^2|w|^2}{I(s)} \int_{\mathbb{R}^m} e^{-s|x|^2}|x|\, d\mathscr{L}^m(x)\,.
\end{aligned}$$

Changing variables in the last integral, recalling that $I(s) = (\pi/s)^{m/2}$, and using Proposition B.1, we obtain

$$\lim_{s\downarrow 0} sI(s)^{-1} \int_{\mathbb{R}^m} e^{-s|x|^2}|x|\, d\mathscr{L}^m(x) = \lim_{s\downarrow 0} C\frac{s^{m/2+1}}{s^{m/2+1/2}} = 0\,.$$

A similar computation yields $b_3^{(2)}(w^3) = 0$ for $w \in V^\perp$ and completes the proof of (9.25).

We now come to (v). Fix $z \in \mathrm{supp}\,(\mu)$. Then from Proposition 7.7 we have

$$b_1^{(2)}(z) + b_2^{(2)}(z^2) + b_3^{(2)}(z^3) + b_4^{(2)}(z^4) = |z|^4\,.$$

Lemma 9.2(i) implies $|P_V(z)|^4 = b_4^{(2)}(z^4)$. Moreover, we have the elementary identity $|z|^4 = |P_V(z)|^4 + 2|P_V(z)|^2|Q_V(z)|^2 + |Q_V(z)|^4$. Hence, we have

$$b_1^{(2)}(z) + b_2^{(2)}(z^2) + b_3^{(2)}(z^3) = |Q_V(z)|^2\big(|Q_V(z)|^2 + 2|P_V(z)|^2\big)\,. \tag{9.27}$$

Moreover, we can write

$$b_3^{(2)}(z^3) = b_3^{(2)}\big((P_V(z) + Q_V(z))^3\big)$$

and using (9.25) we obtain

$$b_3^{(2)}(z^3) = 3b_3^{(2)}\big(P_V(z)^2 \odot Q_V(z)\big)\,.$$

Substituting this into (9.27) we obtain (v).

(vi) Fix $v \in V$. From Lemma 9.4(iv) one can show the existence of $t_i \uparrow \infty$, $v_i \in V$ and $w_i \in V^\perp$ such that

$$t_i v + v_i + w_i \in \mathrm{supp}\,(\mu) \qquad \text{and} \qquad |v_i + w_i| \leq r_0\,.$$

We can assume that, up to subsequences, $w_i \to w$ for some $w \in V^\perp$. Applying (v), we have

$$\begin{aligned}
& b_1^{(2)}(t_i v + v_i + w_i) + b_2^{(2)}((t_i v + v_i + w_i)^2) + 3b_3^{(2)}((t_i v + v_i)^2 \odot w_i) \\
= \;& 2|t_i v + v_i|^2|w_i|^2 + |w_i|^4\,.
\end{aligned}$$

Dividing by t_i^2 and letting $i \uparrow \infty$ we conclude

$$b_2^{(2)}(v^2) + 3b_3^{(2)}(v^2 \odot w) = 2|v|^2|w|^2\,.$$

On the other hand, since $t_i v + v_i + w_i \in \mathrm{supp}\,(\mu)$, from Lemma 9.4(iii) we have

$$b_1^{(1)}(t_i v + v_i + w_i) = |w_i|^2\,.$$

Since $t_i v + v_i \in V$, from Lemma 9.2(ii) we conclude $b_1^{(1)}(t_i v + v_i) = 0$ and hence

$$b_1^{(1)}(w_i) = |w_i|^2\,.$$

Letting $i \uparrow \infty$ we find that $b_1^{(1)}(w) = |w|^2$. Therefore, recalling Definition 9.7 and the fact that $b_1^{(1)}(w) = 2\langle b, w\rangle$, we conclude that

$$\begin{cases} b_2^{(2)}(v^2) + \langle \omega(v^2), w\rangle \;=\; 0 \\ \\ |w|^2 \;=\; 2\langle b, w\rangle\,. \end{cases} \tag{9.28}$$

Set $\hat{v} := w - b$. Then from (9.28) we have

$$|\hat{v}|^2 \;=\; |w|^2 - 2\langle w, b\rangle + |b|^2 \;=\; |b|^2 \tag{9.29}$$

and (recall Definition 9.7)

$$\hat{b}(v^2) + \langle \omega(v^2), \hat{v}\rangle \;=\; 0\,.$$

Therefore, we conclude

$$\Big(\hat{b}(v^2)\Big)^2 \;\leq\; \big|\omega(v^2)\big|^2\,|\hat{v}|^2 \;\stackrel{(9.29)}{=}\; \big|\omega(v^2)\big|^2\,|b|^2\,. \tag{9.30}$$

□

4. Proof of (9.7) and conclusion

We are now ready for the last computations leading to (9.7) and hence to the proofs of Proposition 9.5 and Proposition 9.1.

Proof of the inequality (9.7). **Step 1** Recall the identity (v) of Lemma 9.10

$$0 \;=\; b_1^{(2)}(z) + b_2^{(2)}(z^2) + 3b_3^{(2)}\big((P_V(z))^2 \odot Q_V(z)\big) - |Q_V(z)|^4 - 2|P_V(z)|^2|Q_V(z)|^2 \tag{9.31}$$

Moreover recall the identities

$$|Q_V(z)|^2 \;\stackrel{\text{Lemma 9.4(iii)}}{=}\; b_1^{(1)}(z) \;\stackrel{\text{Lemma 9.2(ii)}}{=}\; b_1^{(1)}(Q_V(z)) \qquad \text{for every } z \in \operatorname{supp}(\mu)\,.$$

Inserting them into (9.31) and using the forms $\hat{b}$ and ω of Definition 9.7, we obtain

$$\begin{aligned} 0 \;=\;& \hat{b}\big((P_V(z))^2\big) + \big\langle \omega\big(P_V(z))^2\big), Q_V(z) - b\big\rangle + b_1^{(2)}(z) \\ &+2b_2^{(2)}\big(P_V(z) \odot Q_V(z)\big) + b_2^{(2)}\big(Q_V(z)^2\big) - \big(b_1^{(1)}(Q_V(z))\big)^2 \qquad \text{for every } z \in \operatorname{supp}(\mu). \end{aligned}$$

From Lemma 9.4(iii) we know that $|Q_V(z)| \leq \|b_1^{(1)}\| = 2|b|$. Therefore, there exists a constant K which gives the following linear growth bound

$$\Big|\hat{b}\big((P_V(z))^2\big) + \big\langle \omega\big((P_V(z))^2\big), Q_V(z) - b\big\rangle\Big| \;\leq\; K\big(|z| + 1\big) \qquad \text{for every } z \in \operatorname{supp}(\mu).$$

Hence, from Lemma 7.2 and Proposition B.1 we conclude

$$\begin{aligned} &\limsup_{s\downarrow 0}\left|\frac{s}{I(s)}\int e^{-s|z|^2}\Big[\hat{b}\big((P_V(z))^2\big) + \big\langle \omega\big((P_V(z))^2\big), Q_V(z) - b\big\rangle\Big]\, d\mu\right| \\ \leq\;& K\lim_{s\downarrow 0}\frac{s}{I(s)}\int e^{-s|z|^2}\big(|z|+1\big)\, d\mu(z) \\ =\;& K\lim_{s\downarrow 0}\frac{s}{I(s)}\int_{\mathbb{R}^m} e^{-s|x|^2}\big(|x|+1\big)\, d\mathscr{L}^m(x) \;=\; 0\,. \end{aligned} \tag{9.32}$$

Step 2 From Proposition B.1 we compute

$$\int_V \langle \zeta, v\rangle^4 d\gamma(v) \;=\; 3|\zeta|^4 \qquad \text{for every } \zeta \in V. \tag{9.33}$$

Indeed, fix orthonormal coordinates $x_1, \ldots, x_m$ on V in such a way that $\zeta = (|\zeta|, 0, \ldots, 0)$. From the definition of γ we obtain

$$\begin{aligned}
\int_V \langle \zeta, v\rangle^4 d\gamma(v) &= (2\pi)^{-m/2}\int_{\mathbb{R}^m} |\zeta|^4 x_1^4 e^{-|x|^2/2}\,dx \;=\; |\zeta|^4\pi^{-m/2}\int_{\mathbb{R}^m} 4y_1^4 e^{-|y|^2}\,dy\\
&= 4|\zeta|^4\pi^{-m/2}\left(\int_{\mathbb{R}^{m-1}} e^{-|y'|^2}\,dy'\right)\cdot\left(\int_{\mathbb{R}} y_1^4 e^{-y_1^2}\,dy_1\right)\\
&= 4|\zeta|^4\pi^{-m/2}\left(\pi^{(m-1)/2}\right)\left(\frac{3}{4}\pi^{-1/2}\right) \;=\; 3|\zeta|^4 .
\end{aligned}$$

Similarly, if $y, z \in V$ are orthogonal, we fix orthonormal coordinates $x_1, \ldots, x_m$ on V so that $y = (|y|, 0, \ldots, 0)$ and $z = (0, |z|, \ldots, 0)$ and we obtain

$$\begin{aligned}
\int_V \langle y, v\rangle^2\langle z, v\rangle^2 d\gamma(v) &= 4|y|^2|z|^2\pi^{-m/2}\left(\int_{\mathbb{R}^{m-2}} e^{-|y'|^2}\,dy'\right)\\
&\quad\cdot\left(\int_{\mathbb{R}} y_1^2 e^{-y_1^2}\,dy_1\right)\cdot\left(\int_{\mathbb{R}} y_2^2 e^{-y_2^2}\,dy_2\right) \;=\; |y|^2|z|^2
\end{aligned} \tag{9.34}$$

and

$$\begin{aligned}
\int_V \langle y, v\rangle\langle z, v\rangle^3 d\gamma(v) &= 4|y||z|^2\pi^{-m/2}\left(\int_{\mathbb{R}^{m-2}} e^{-|y'|^2}\,dy'\right)\\
&\quad\cdot\left(\int_{\mathbb{R}} y_1 e^{-y_1^2}\,dy_1\right)\cdot\left(\int_{\mathbb{R}} y_2^3 e^{-y_2^2}\,dy_2\right) \;=\; 0\,.
\end{aligned} \tag{9.35}$$

For general $y, z \in V$, we write $y = \xi + az$, where $\xi \perp \zeta$ and we compute

$$\begin{aligned}
\int_V \langle y, v\rangle^2\langle z, v\rangle^2 d\gamma(v) &= \int_V \langle \xi, v\rangle^2\langle z, v\rangle^2\,d\gamma(v) + 2a\int_V \langle \xi, v\rangle\langle z, v\rangle^3\,d\gamma(v)\\
&\quad + a^2\int_V \langle z, v\rangle^4\,d\gamma(v) \overset{(9.34)+(9.35)}{=} |\xi|^2|z|^2 + 3a^2|z|^4\\
&= (|\xi|^2 + a^2|z|^2)|z|^2 + 2(a|z|^2)^2 \;=\; |y|^2|z|^2 + 2\langle z, y\rangle^2 .
\end{aligned} \tag{9.36}$$

Now we fix $y \in V$ and $w \in V^\perp$ and we compute

$$\begin{aligned}
&\int_V \langle y, v\rangle^2\langle \omega(v^2), w\rangle\,d\gamma(v)\\
\overset{(9.18)}{=}\;& \lim_{s\downarrow 0}\frac{8s}{I(s)}\int e^{-s|z|^2}\langle z-b, w\rangle\left[\int_V \langle y, v\rangle^2\langle z, v\rangle^2\,d\gamma(v)\right]\,d\mu(z)\\
\overset{(9.36)}{=}\;& \lim_{s\downarrow 0}\frac{16s}{I(s)}\int e^{-s|z|^2}\langle z-b, w\rangle\langle z, y\rangle^2\,d\mu(z)\\
&+\lim_{s\downarrow 0}\frac{8s|y|^2}{I(s)}\int e^{-s|z|^2}|P_V(z)|^2\langle z-b, w\rangle\,d\mu(z)\\
\overset{(9.23)+(9.18)}{=}\;& 2\langle \omega(y^2), w\rangle .
\end{aligned} \tag{9.37}$$

Therefore, we have

$$\begin{aligned}\int_V \left(\hat{b}(v^2)\right)^2 d\gamma(v) &\overset{(9.24)}{\leq} |b|^2 \int_V \left|\omega(v^2)\right|^2 d\gamma(v)\\ &\overset{(9.18)}{=} |b|^2 \lim_{s\downarrow 0} \int_V \frac{8s}{I(s)} \int e^{-s|z|^2} \langle z, v\rangle^2 \langle z-b, \omega(v^2)\rangle \, d\mu(z)\, d\gamma(v)\\ &= |b|^2 \lim_{s\downarrow 0} \frac{8s}{I(s)} \int e^{-s|z|^2} \int_V \langle z, v\rangle^2 \langle z-b, \omega(v^2)\rangle \, d\gamma(v)\, d\mu(z)\,.\end{aligned}$$

Recall that in Remark 9.8 we noticed that the values of ω are all contained in $V^\perp$. Thus $\langle z-b, \omega(v^2)\rangle = \langle Q_V(z)-b, \omega(v^2)\rangle$. Moreover, since $\mathrm{supp}\,(\gamma) = V$, we can write

$$\begin{aligned}\int_V \left(\hat{b}(v^2)\right)^2 d\gamma(v) &= |b|^2 \lim_{s\downarrow 0} \frac{8s}{I(s)} \int e^{-s|z|^2} \int_V \langle P_V(z), v\rangle^2 \langle Q_V(z)-b, \omega(v^2)\rangle \, d\gamma(v)\, d\mu(z)\\ &\overset{(9.37)}{=} |b|^2 \lim_{s\downarrow 0} \frac{16s}{I(s)} \int e^{-s|z|^2} \left\langle \omega\big((P_V(z))^2\big), Q_V(z)-b\right\rangle \, d\mu(z)\\ &\overset{(9.32)}{=} -16|b|^2 \lim_{s\downarrow 0} \frac{s}{I(s)} \int e^{-s|z|^2} \hat{b}\big((P_V(z))^2\big)\, d\mu(z)\\ &= -16\pi^{-m/2}|b|^2 \int e^{-|z|^2} \hat{b}\big((P_V(z))^2\big) d\mathscr{H}^m \llcorner V(z)\,, \end{aligned} \tag{9.38}$$

where in the last line we have used that $\mathscr{H}^m \llcorner V$ is tangent to μ at infinity.

After a change of variables, we conclude

$$\int_V \left(\hat{b}(v^2)\right)^2 d\gamma(v) \;\leq\; -8|b|^2 \int_V \hat{b}(v^2)\, d\gamma(v)\,. \tag{9.39}$$

Step 3 Let $\beta_1, \ldots, \beta_m$ be the eigenvalues of $\hat{b}$ and fix coordinates $v_1, \ldots, v_m$ on V in such a way that the unit vectors $e_1, \ldots, e_m$ are the eigenvectors of $\hat{b}$. Then we have

$$\begin{aligned}\int_V \hat{b}(v^2)\, d\gamma(v) &= \int_V \beta_1 v_1^2 + \ldots + \beta_m v_m^2 \, d\gamma(v)\\ &= \beta_1 + \ldots + \beta_m \;=\; \mathrm{tr}\, \hat{b}\end{aligned} \tag{9.40}$$

and

$$\begin{aligned}\int_V \left(\hat{b}(v^2)\right)^2 d\gamma(v) &= \int_V \left(\beta_1 v_1^2 + \ldots + \beta_m v_m^2\right)^2 d\gamma(v)\\ &= \sum_{i=1}^m \beta_i^2 \int_V v_i^4 \, d\gamma(v) + 2\sum_{j>i} \beta_j\beta_i \int_V v_i^2 v_j^2 \, d\gamma(v)\,.\end{aligned}$$

Recalling (9.33) and (9.34) we obtain

$$\begin{aligned}\int_V \left(\hat{b}(v^2)\right)^2 d\gamma(v) &= 3\sum_{i=1}^m \beta_i^2 + 2\sum_{j>i} \beta_j\beta_i \;=\; \left(\sum_{i=1}^m \beta_i\right)^2 + 2\sum_{i=1}^m \beta_i^2\,.\\ &\geq \left(1+\frac{2}{m}\right)\left(\sum_{i=1}^m \beta_i\right)^2 \;=\; \left(1+\frac{2}{m}\right)\left[\int_V \hat{b}(v^2)\, d\gamma(v)\right]^2.\end{aligned}$$

Therefore, from (9.39) we conclude

$$\left(1+\frac{2}{m}\right)\left[\int_V \hat{b}(v^2)\,d\gamma(v)\right]^2 \;\le\; -8|b|^2\int_V \hat{b}(v^2)\,d\gamma(v)$$

and hence

$$\int_V \hat{b}(v^2)\,d\gamma(v) \;\ge\; -\left(1+\frac{2}{m}\right)^{-1} 8|b|^2 \;=\; -\frac{2m}{m+2}\left|b_1^{(1)}\right|^2 . \tag{9.41}$$

Combining (9.41) with Lemma 9.9 we obtain (9.7). □

Proofs of Proposition 9.1 and of Proposition 9.5. Concerning Proposition 9.5, note that (9.3) is proved in the introduction of the chapter, whereas (9.4) follows from (9.6) (which is also proved in the introduction of the chapter) and (9.7).

Coming to Proposition 9.1, note that (9.3) and (9.4) give $b_1^{(1)} = 0$. By Lemma 9.4(iii) this implies $\operatorname{supp}(\mu) \subset V$. As already remarked upon many times, since μ is an m–uniform measure and V an m–dimensional plane, this implies that $\mu = \mathscr{H}^m \llcorner V$, which is the desired conclusion. □

CHAPTER 10

Open problems

This chapter contains several open problems related to the topics of these notes, which I collected with the help of Bernd Kirchheim.

0.1. Lower and upper densities and Besicovitch's $1/2$–**Conjecture.** As already mentioned in the introduction, the Theorem proved by Preiss in [**25**] is stronger than the one exposed in the second part of these notes. We recall it here for the reader's convenience (cp. with Theorem 1.2).

THEOREM 10.1. *For any pair of nonnegative integers $k \leq n$ there exists a constant $c(k,n) > 1$ such that the following holds. If μ is a locally finite measure on $\mathbb{R}^n$ and*

$$0 \;<\; \limsup_{r\downarrow 0} \frac{\mu(B_r(x))}{r^k} \;<\; c(k,n) \liminf_{r\downarrow 0} \frac{\mu(B_r(x))}{r^k} \;<\; \infty \qquad \text{for } \mu\text{–a.e. } x \in \mathbb{R}^n, \tag{10.1}$$

then μ is a rectifiable k–dimensional measure.

The following is an outstanding open problem.

PROBLEM 10.2. *What are the optimal constants $c(k,n)$ for which Theorem 10.1 holds? How do they behave for $n \uparrow \infty$?*

Very little is known in this direction. In his paper, Preiss shows that $c(2,n)$ converges to 0 as $n \uparrow \infty$. There is a striking difference with the case $k = 1$: Moore proved in [**23**] that $c(1,n) \geq 1 + 1/100$ for every n. This fact gives a glimpse of why the case $k \geq 2$ of Theorem 1.2 is much more difficult than the case $k = 1$.

A natural interesting case of Problem 10.2 is given by measures μ of the form $\mathscr{H}^k \llcorner E$ for some Borel set E. In this case the upper density $\theta^{k*}(\mu,x)$ is necessarily less or equal than 1 at μ–almost every point x. Therefore, as a corollary of Theorem 10.1 we conclude that

COROLLARY 10.3. *For any pair of nonnegative integers $k \leq n$ let $c(k,n) > 1$ be the constants of Theorem 10.1. Then, any Borel set E with $0 < \mathscr{H}^k(E) < \infty$ such that*

$$\alpha(k,n) \;:=\; c(k,n)^{-1} \;<\; \theta^k_*(E,x) \qquad \text{for } \mathscr{H}^k\text{–a.e. } x \in E \tag{10.2}$$

is a rectifiable set.

Thus we can ask the following question.

PROBLEM 10.4. *What are the optimal constants $\alpha(k,n)$ under which Corollary 10.3 hold?*

Though clearly $\alpha(k,n) \leq [c(k,n)]^{-1}$, it is not known whether one can control $\alpha(k,n)$ from below with $c(k,n)$. The optimal constant $\alpha(1,n)$ was conjectured long ago in [**2**]. This is the famous Besicovitch $1/2$–Conjecture:

CONJECTURE 10.5. *If $E \subset \mathbb{R}^2$ is a Borel set with $0 < \mathscr{H}^1(E) < \infty$ and $\theta^1_*(E,x) > 1/2$ for $\mathscr{H}^1$–a.e. x, then E is rectifiable.*

In his seminal paper [**2**] Besicovitch proved the bound $\alpha(1,2) \leq 3/4$. His proof generalizes easily to show that $\alpha(1,n) \leq 3/4$ for every n. The bound $\alpha(1,2) \geq 1/2$ was already proved by Besicovitch in [**2**] (see also [**5**]). More precisely he exhibited a purely unrectifiable set E which has lower density equal to $1/2$ $\mathscr{H}^1$–almost everywhere. Besicovitch's estimate $\alpha(1,n) \leq 3/4$ remained for long time the best until Preiss and Tišer in [**26**] improved it to $\alpha(1,n) \leq (2+\sqrt{46})/12$. A more important feature of their proof is that it actually extends to general metric spaces. Recent attempts to solve Besicovitch's Conjecture can be found in [**6**] and [**7**]. Concerning the value of the optimal constant $c(k,n)$ for general k and n very little is known. In [**3**] Chlebík proved that $\alpha(k,n) \geq 1/2$.

An "ε"–version of Marstrand's Theorem 3.1 is valid as well. More precisely the following theorem holds and its proof is a routine application of the techniques introduced in Chapter 3.

THEOREM 10.6. *Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$ and $n \in \mathbb{N}$. Then there exists a positive constant $\varepsilon(\alpha,n)$ such that*

$$\mu\left(\left\{x : 0 < \limsup_{r\downarrow 0} \frac{\mu(B_r(x))}{r^\alpha} \leq (1+\varepsilon(\alpha,n)) \liminf_{r\downarrow 0} \frac{\mu(B_r(x))}{r^\alpha}\right\}\right) = 0\,, \tag{10.3}$$

for every measure μ.

PROOF. Let $\varepsilon > 0$ be a fixed positive number. Arguing with a blow–up procedure as in the proof of Marstrand's Theorem 3.1, in order to show the conclusion of the theorem for this particular ε, it suffices to show that there are no non–trivial measures ν such that

$$\omega_\alpha r^\alpha \leq \nu(B_r(x)) \leq (1+\varepsilon)\omega_\alpha r^\alpha \qquad \text{for every } x \in \operatorname{supp}(\mu) \text{ and every } r > 0. \tag{10.4}$$

So, if the theorem were false, for every $\varepsilon > 0$ there would exist a measure ν_ε with $0 \in \operatorname{supp}(\nu_\varepsilon)$ satisfying the bounds (10.4). By compactness, a subsequence of $\{\nu_\varepsilon\}_{\varepsilon\downarrow 0}$ converges to a measure $\nu_0 \in \mathcal{U}^\alpha(\mathbb{R}^n)$. But we know from Proposition 3.5 that $\mathcal{U}^\alpha(\mathbb{R}^n)$ is empty for every $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$. This gives a contradiction and concludes the proof. □

As above, we can ask

PROBLEM 10.7. *What are the optimal constants $\varepsilon(\alpha,n)$ for which Theorem 10.6 holds? How do they behave as $n \uparrow \infty$?*

Even the following question is yet unsolved.

PROBLEM 10.8. *Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$. Does $\liminf_n \varepsilon(\alpha,n) > 0$?*

0.2. Noneuclidean setting. Another outstanding problem is to extend the validity of Theorem 1.1 to more general geometries. In particular the following natural conjecture (see for instance [**4**]) is widely open.

CONJECTURE 10.9. *Let X be a finite–dimensional Banach space, α a non–integer positive number and μ a measure on X such that*

$$0 < \lim_{r\downarrow 0} \frac{\mu(B_r(x))}{r^\alpha} < \infty \qquad \text{for } \mu\text{–a.e. } x \in X, \tag{10.5}$$

where $B_r(x)$ denotes the intrinsic ball of radius r and center x. Then $\mu = 0$.

For $\alpha\in]0,1[$ the answer to the Conjecture is affirmative and follows from a metric version of the arguments of Marstrand in ([**15**]). The following much more challenging case of Conjecture 10.9, recently proved by Lorent in [**14**], is the only other known extension of Marstrand's result.

THEOREM 10.10. *Conjecture 10.9 holds for $\alpha\in]1,2[$ if the balls of X are polytopes.*

EXAMPLE 10.11. *It is not difficult to see that Conjecture 10.9 does not hold for gemeral metric groups X. For instance one might take $X=\mathbb{R}^1$ with distance $d(x,y)=|y-x|^{1/\alpha}$ and the measure $\mu=\mathscr{H}^\alpha$, for $\alpha\in]1,2[$ (cp. with [**4**]).*

A natural generalization of Preiss' rectifiability theorem is the following.

CONJECTURE 10.12. *Let X be a finite–dimensional Banach space, $k\in\mathbb{N}$ a non–integer positive number and μ a measure on X such that*

$$0<\lim_{r\downarrow 0}\frac{\mu(B_r(x))}{r^k}<\infty\qquad\text{for } \mu\text{–a.e. } x\in X. \tag{10.6}$$

Then μ is a rectifiable k–dimensional measure.

For more general metric spaces X this Conjecture fails (take X as in Example 10.11 and choose $\alpha=2$). Clearly the case $k=0$ is trivial. The case $k=1$ follows by a suitable modification of Besicovitch's arguments. For $k\geq 2$ the validity of Conjecture 10.12 is known only in the Euclidean space. Even the following stronger version is open:

CONJECTURE 10.13. *Let X be a finite–dimensional Banach space, $k\geq 2$ an integer and E a Borel set of $\mathbb{R}^n$ with finite $\mathscr{H}^k$–measure such that*

$$\lim_{r\downarrow 0}\frac{\mathscr{H}^k(E\cap B_r(x))}{\omega_k r^k}\;=\;1\qquad\text{for } \mathscr{H}^k\text{–a.e. } x\in E. \tag{10.7}$$

Then E is a rectifiable k–dimensional set.

In the model case $X=\ell^3_\infty$ (i.e., $\mathbb{R}^3$ with the cube norm) Lorent carried out a considerable part of Preiss' strategy in [**12**]. The situation here is much more complicated because of the large abundance of uniform measures (see the next section for a related problem).

0.3. Uniform measures. We start this section by defining uniform measures, a suitable generalization of k–uniform measures.

DEFINITION 10.14. *A locally finite measure μ on $\mathbb{R}^n$ is said to be a uniform measure if for every $r>0$ and for every $x,y\in\operatorname{supp}(\mu)$ we have $\mu(B_r(x))=\mu(B_r(y))$.*

The following elegant theorem of Kirchheim and Preiss (see [**10**] and Theorem 3.11) proves a strong regularity property of uniform measures (the proof is reported in Appendix A).

THEOREM 10.15. *If μ is a uniform measure, then $\operatorname{supp}(\mu)$ is a real analytic variety, i.e. there exists an analytic function $H:\mathbb{R}^n\to\mathbb{R}$ such that $\operatorname{supp}(\mu)=\{H=0\}$.*

A standard stratification result shows that analytic varieties are the union of finitely many strata, each of which is an analytic submanifold of integer dimension. If k is the dimension of the "top" stratum, then $\operatorname{supp}(\mu)$ is a k–dimensional rectifiable set, and it is easy to check that $\mu=c\mathscr{H}^k\llcorner\operatorname{supp}(\mu)$ for some positive constant c.

The following is a very natural and hard problem.

PROBLEM 10.16. *Classify all uniform measures.*

Even the very particular case of classifying all discrete uniform measures is open.

Clearly k–uniform measures are a particular example of uniform measures, and hence the following is another very particular case of Problem 10.16

PROBLEM 10.17. *Give a complete description of* $\mathcal{U}^k(\mathbb{R}^m)$ *for every pair of integers* k *and* m.

A solution of this last classification problem would yield a different point of view on Preiss' proof, i.e. a deeper understanding of why flat uniform measures and uniform measures curved at infinity form a disconnected set.

The case $k = m - 1$ of Problem 10.17 has been settled by Kowalski and Preiss in [**11**].

THEOREM 10.18. $\mu \in \mathcal{U}^{m-1}(\mathbb{R}^m)$ *if and only if* μ *is flat or* $m \geq 4$ *and there exists an orthonormal system of coordinates such that* $\mu = \mathscr{H}^{m-1} \llcorner C$, *where* C *is the cone* $\{x_1^2 + x_2^2 + x_3^2 = x_4^2\}$.

Flat measures and measures of the form $\mathscr{H}^k \llcorner V \times C$ (where $V \times C \subset \mathbb{R}^{n-4} \times \mathbb{R}^4$ is the product of a linear subspace of V with the light cone C) are the only known examples of k–uniform measures. Therefore even the following seemingly innocent question is still open:

QUESTION 10.19. *Are there* k*–uniform measures which are nor flat measures nor products between the light cone and flat measures?*

A natural way of constructing uniform measures is to look for k–dimensional homogeneous sets $Z \subset \mathbb{R}^n$.

DEFINITION 10.20. *A set* $Z \subset \mathbb{R}^n$ *is homogeneous if for every* $x, y \in Z$ *there exists an isometry* Φ *of* $\mathbb{R}^n$ *such that* $\Phi(x) = y$ *and* $\Phi(Z) = Z$.

One could naively conjecture that all uniform measures are homoegenous, i.e. of the type $c\mathscr{H}^k \llcorner Z$ for an homogeneous set, but the light cone shows once more that this is not the case. However the following questions are still open

QUESTION 10.21. *Are there non–homogeneous uniform measures in* $\mathbb{R}^n$ *for* $n \leq 3$*?*

QUESTION 10.22. *Are there non–homogeneous uniform measures with bounded support?*

QUESTION 10.23. *Are there non–homogeneous uniform measures with discrete support?*

Conjecture 10.12 leads naturally to the study of measures which are m–uniform with respect to different geometries. As already mentioned, the case of ℓ^3_∞ has a large abundance of 2–uniform measures. For instance, if $\Gamma = \{(x_1, x_2, f(x_1, x_2)\}$ is the graph of a 1–Lipschitz function $f : \ell^2_\infty \to \mathbb{R}$, then the measure $\mathscr{H}^2 \llcorner \Gamma$ is a 2–uniform measures (here $\mathscr{H}^2$ denotes the Hausdorff 2–dimensional measure relative to the metric space ℓ^3_∞). On the other hand $\mathscr{H}^2 \llcorner V$ is a 2–uniform measure for any linear 2–dimensional subspace V. The following is a plausible conjecture.

CONJECTURE 10.24. *Let* $\mathscr{H}^2 \llcorner Z$ *be a* 2*–uniform measure in* ℓ^3_∞. *Then either* Z *is a linear subspace, or it is the graph of a* 1*–Lipschitz function* $f : \ell^3_\infty \to \mathbb{R}$, *up to a permutation of* x_1, x_2, x_3.

0.4. Exact density functions. Let us introduce the following terminology.

DEFINITION 10.25. *A function $h : \mathbb{R}^+ \to \mathbb{R}^+$ is called an exact density function in $\mathbb{R}^n$ if there exists a nontrivial measure μ on $\mathbb{R}^n$ such that*

$$0 < \lim_{r\downarrow 0} \frac{\mu(B_r(x))}{h(r)} < \infty \qquad \text{for } \mu\text{–a.e. } x. \tag{10.8}$$

Therefore Marstrand's Theorem 3.1 can be restated in the following way.

THEOREM 10.26. *$h(r) = r^\alpha$ is an exact density function in $\mathbb{R}^n$ if and only if α is a natural number less or equal than n.*

It has been proved by Mattila in [**19**] that in 1 dimension an exact density function must satisfy the conditions

$$0 < \lim_{r\downarrow 0} h(r) < \infty \qquad \text{or} \qquad 0 < \lim_{r\downarrow 0} \frac{h(r)}{r} < \infty\,. \tag{10.9}$$

Section 6 of [**25**] contains several results and questions about exact density functions and more complicated variants of them.

0.5. Symmetric measures and singular integrals. Some of the ideas of [**25**] have been used by Mattila and Preiss in [**22**] to prove the following rectifiability result (see also the previous work [**20**] of Mattila in two dimensions).

THEOREM 10.27. *Let μ be a finite Borel measure on $\mathbb{R}^n$ such that $0 < \theta^k_*(\mu, x) < \infty$ for μ–a.e. x. If the principal value*

$$\lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{y-x}{|y-x|^{k+1}}\, d\mu(y) \tag{10.10}$$

exists at μ–a.e. x, then μ is a rectifiable k–dimensional measure.

In [**22**] the authors raised the following question

QUESTION 10.28. *Assume that $\mu = \mathscr{H}^k \llcorner E$ for some Borel set E with finite $\mathscr{H}^k$ measure and that* (10.10) *exists for μ–a.e. x. Is it then possible to drop the lower density assumption $\theta^k_*(\mu, x) > 0$ in Theorem 10.27?*

If we replace (10.10) by the existence of the principal value for other singular kernels of type $\Omega(x/|x|)|x|^{-k}$ with Ω odd, then the conclusion of Theorem 10.27 is false (see [**9**]). It is an open problem to understand for which type of kernels one can generalize the rectifiability result of Mattila and Preiss.

The proof of Theorem 10.27 uses a blow–up technique and a carefuly study of the tangent measures to μ. In particular, using this approach, one ends up studying symmetric measures.

DEFINITION 10.29. *A measure ν on $\mathbb{R}^n$ is called k–*dimensional symmetric measure *if*

$$\int_{B_r(x)} (z-x)\, d\nu(z) \;=\; 0 \qquad \text{for every } x \in \operatorname{supp}(\nu) \text{ and any } r > 0, \tag{10.11}$$

and there exists a positive constant c such that

$$\nu(B_r(x)) \geq c r^k > 0 \qquad \text{for every } x \in \operatorname{supp}(\nu). \tag{10.12}$$

In [**20**] Mattila showed that in $\mathbb{R}^2$ symmetric measures are necessarily sums of flat measures. In higher dimension the question whether a similar result holds is open.

APPENDIX A

Proof of Theorem 3.11

Before coming to the proof we recall that the assumption (3.6) implies the following identity for every $y, z \in \operatorname{supp}(\mu)$ and any μ–summable radial function $\varphi(|\cdot|)$:

$$\int \varphi(|x-y|)\, d\mu(x) \;=\; \int \varphi(|x-z|)\, d\mu(x) \tag{A.1}$$

(see Remark 3.15 and the proof of Lemma 7.2).

Proof. In order to simplify the notation, from now on we denote $\operatorname{supp}(\mu)$ by S and, given any $x \in S$, we introduce the function $f : \mathbb{R}^+ \to \mathbb{R}$ given by

$$f(s) \;:=\; \mu(B_s(x))\,.$$

By (A.1), f does not depend on the choice of $x \in S$.

Step 1 In this step we prove the following estimate:

$$\mu(B_r(y)) \;\le\; 5^n \left(\frac{r}{s}\right)^n f(s) \qquad \text{for every } 0 < s < r < \infty \text{ and for every } y \in \mathbb{R}^n. \tag{A.2}$$

Denote by $\mathcal{F}$ the family of finite subsets $Z \subset B_r(y)$ such that

$$|z_1 - z_2| \;\ge\; \frac{s}{2} \qquad \text{for every } z_1, z_2 \in Z \text{ with } z_1 \neq z_2.$$

Fix $Z \in \mathcal{F}$ and note that the balls $\{B_{s/4}(z) : z \in Z\}$ are all disjoint and contained in $B_{5/4r}(y)$. Therefore

$$\operatorname{card}(Z)\, \omega_n \left(\frac{s}{4}\right)^n \;=\; \sum_{z \in Z} \mathscr{L}^n(B_{s/4}(z)) \;\le\; \mathscr{L}^n(B_{5r/4}(y)) \;=\; \omega_n \left(\frac{5r}{4}\right)^n$$

and we conclude that

$$\operatorname{card}(Z) \;\le\; 5^n (r/s)^n\,. \tag{A.3}$$

This allows us to choose a set $M \in \mathcal{F}$ such that $\operatorname{card}(M) = \max_{Z \in \mathcal{F}} \operatorname{card}(Z)$.

Fix $z \in M$. Then, either $B_{s/2}(z) \cap S = \emptyset$ and hence $\mu(B_{s/2}(z)) = 0$, or there exists $y \in B_{s/2}(z) \cap S$ and thus $\mu(B_{s/2}(z)) \le \mu(B_s(y)) = f(s)$. In any case,

$$\mu(B_{s/2}(z)) \;\le\; \mu(B_s(x)) \;=\; f(s) \qquad \text{for any } z \in M. \tag{A.4}$$

Note that the maximality of M implies that the balls $\{B_{s/2}(y)\}_{y \in M}$ cover $B_r(x)$. Thus

$$\mu(B_r(x)) \;\le\; \sum_{y \in M} \mu\left(B_{s/2}(y)\right) \;\overset{\text{(A.4)}}{\le}\; \operatorname{card}(Z) f(s) \;\overset{\text{(A.3)}}{\le}\; 5 \left(\frac{r}{s}\right)^n f(s)\,.$$

Step 2 Let us fix $x^0 \in S$ and set

$$F(x,s) \; := \; \int \left[e^{-s|z-x|^2} - e^{-s|z-x^0|^2} \right] d\mu(z) \qquad \text{for } x \in \mathbb{R}^n \text{ and } s>0. \tag{A.5}$$

For any $x \in \mathbb{R}^n$ we can write

$$\int e^{-s|z-x|^2} \, d\mu(z) \; = \; \int \mu\left(\left\{ z : e^{-s|z-x|^2} < r \right\}\right) dr \; = \; \int_0^1 \mu\left(B_{\sqrt{-(\ln r)/s}}(x)\right) dr\,. \tag{A.6}$$

Therefore:

- From (A.2) and (A.6) we conclude

$$\int e^{-s|z-x|^2} \, d\mu(z) \; \le \; 5^n f\left(s^{-1/2}\right) \int_0^1 (-\ln r)^{n/2} \, dr \; < \; \infty \tag{A.7}$$

  and thus the integral in (A.5) is finite.
- From (A.1) we conclude that the integral in (A.5) vanishes for any $x \in \operatorname{supp}(\mu)$.

Therefore, F is well defined, finite, and $F(x,s) = 0$ for any $s>0$ and $x \in S$. We claim that

$$F(x,s) \; = \; 0 \quad \text{for any } s > 1 \qquad \Longleftrightarrow \qquad x \; \in \; S\,. \tag{A.8}$$

As we have already remarked, one implication follows directly from (A.1). It remains to prove the opposite implication. Assume that $x \notin S$ and let $\varepsilon > 0$ be such that $B_\varepsilon(x) \cap S = \emptyset$. Then we have

$$\begin{aligned} \int e^{-s|z-x|^2} \, d\mu(z) \quad &= \quad \sum_{i=1}^{\infty} \int_{B_{(k+1)\varepsilon}(x) \setminus B_{k\varepsilon}(x)} e^{-s|z-x|^2} \, d\mu(z) \; \le \; \sum_{i=1}^{\infty} e^{-sk^2\varepsilon^2} \mu\big(B_{(k+1)\varepsilon}(x)\big) \\ &\overset{(A.2)}{\le} \quad 10^n \sum_{k=1}^{\infty} e^{-sk^2\varepsilon^2} (k+1)^n f(\varepsilon/2) \end{aligned} \tag{A.9}$$

and

$$\int e^{-s|z-x^0|^2} \, d\mu(z) \; \ge \; e^{-s\varepsilon^2/4} \mu\big(B_{\varepsilon/2}(x^0)\big) \; = \; e^{-s\varepsilon^2/4} f(\varepsilon/2)\,. \tag{A.10}$$

These inequalities imply that

$$\lim_{s \uparrow \infty} \frac{\int e^{-s|z-x|^2} \, d\mu(z)}{\int e^{-s|z-x^0|^2} \, d\mu(z)} \; \le \; \lim_{s \uparrow \infty} 10^n \sum_{k=1}^{\infty} e^{-s(k^2-1/4)\varepsilon^2} (k+1)^n \; = \; 0\,.$$

Therefore, for s large enough, $F(x,s)$ must be negative.

Step 3 We now define

$$H(x) \; := \; \int_1^\infty e^{-s^2} F^2(x,s) \, ds\,. \tag{A.11}$$

Recalling (A.7), we have

$$\begin{aligned} F^2(x,s) &\le 2\left(\int e^{-s|z-x|^2}\,d\mu(x)\right)^2 + 2\left(\int e^{-s|z-x^0|^2}\,d\mu(x)\right)^2 \\ &\le 2\cdot 5^{2n}[f(s^{-1/2})]^2\left(\int_0^1 (-\ln r)^{n/2}\,dr\right)^2 \\ &\le 2\cdot 5^{2n}[f(1)]^2\left(\int_0^1 (-\ln r)^{n/2}\,dr\right)^2 \qquad \text{for } s>1\,. \end{aligned} \tag{A.12}$$

Thus H is finite and moreover, by (A.8), $H(x)=0$ if and only if $x\in S$. To complete the proof of the theorem we just need to show that H is analytic.

First of all, note that H can be extended to a complex function of $\mathbb{C}^n$ by simply setting

$$H(\xi_1,\dots,\xi_n) := \int_1^\infty e^{-s^2}\left[\int\left(e^{-s\sum_j(z_j-\xi_j)^2} - e^{-s|z-x^0|^2}\right)d\mu(z)\right]^2 ds \tag{A.13}$$

for every $(\xi_1,\dots,\xi_n)\in\mathbb{C}^n$. We will now show that this extension is holomorphic.

First of all, set

$$h(s,z,\xi) := e^{-s\sum_j(z_j-\xi_j)^2} - e^{-s|z-x^0|^2}\,.$$

Thus, we can write

$$H(\xi) = \int_1^\infty e^{-s^2}\left[\int h(s,z,\xi)\,d\mu(z)\right]^2 ds\,.$$

Next note that

$$h(s,z,\xi) = e^{-s|z-\operatorname{Re}\xi|^2+s|\operatorname{Im}\xi|^2+2si(z-\operatorname{Re}\xi)\cdot\operatorname{Im}\xi} - e^{-s|z-x^0|^2}\,,$$

from which we obtain

$$|h(s,z,\xi)| \le e^{-s|z-x^0|^2} + e^{s|\operatorname{Im}\xi|^2}e^{-s|z-\operatorname{Re}\xi|^2}\,. \tag{A.14}$$

We use this inequality to estimate

$$\begin{aligned} \left|\int h(s,z,\xi)\,d\mu(z)\right| &\le \int |h(s,z,\xi)|\,d\mu(z) \\ &\le e^{s|\operatorname{Im}\xi|^2}\int e^{-s|z-\operatorname{Re}\xi|^2}\,d\mu(z) + \int e^{-s|z-x^0|^2}\,d\mu(z) \\ &\overset{(A.7)}{\le} C\left(1+e^{s|\operatorname{Im}\xi|^2}\right) \qquad \text{for } s>1. \end{aligned} \tag{A.15}$$

This gives

$$|H(\xi)| \le C\int e^{-s^2}\left(1+e^{2s|\operatorname{Im}\xi|^2}\right)ds < \infty\,.$$

Next note that we have

$$\frac{\partial h}{\partial\bar\xi_j}(s,z,\xi) = 0 \qquad \text{for every } j. \tag{A.16}$$

Fix a direction $\omega\in\mathbb{C}^n$. We want to show that $\frac{\partial H}{\partial\omega}$ exists at every ξ and that

$$\frac{\partial H}{\partial\omega}(\xi) = \int_1^\infty e^{-s^2}2\left[\int h(s,z,\xi)\,d\mu(z)\int\frac{\partial h}{\partial\omega}(s,z,\xi)\,d\mu(z)\right]ds\,. \tag{A.17}$$

This, together with (A.16), would imply that H is holomorphic and completes the proof.

Therefore, fix $\xi, \omega \in \mathbb{C}^n$ and consider

$$\begin{aligned}\lim_{t\in\mathbb{R},\, t\downarrow 0} \frac{H(\xi+t\omega)-H(\xi)}{t} &= \lim_{t\in\mathbb{R},\, t\downarrow 0} \int_1^\infty e^{-s^2} \left[\int \big(h(s,z,\xi+t\omega) + h(s,z,\xi) \big)\, d\mu(z) \right. \\ &\qquad \left. \cdot \int \frac{h(s,z,\xi+t\omega)-h(s,z,\xi)}{t}\, d\mu(z) \right] ds\,.\end{aligned}$$

Recalling (A.14), for fixed ξ and ω, and for $t \le 1$ we obtain

$$\begin{aligned} f_{s,\xi,,\omega,t}(z) &:= |h(s,z,\xi+t\omega)+h(s,z,\xi)| \\ &\le Ce^{Cs}\left(e^{-s|z-x_0|^2} + e^{-s|z-\mathrm{Re}\,(\xi+t\omega)|^2} + e^{-s|z-\mathrm{Re}\,\xi|^2} \right).\end{aligned}$$

Therefore, it is easy to see that there exists $f_{s,\xi,\omega} \in L^1(\mu)$ such that $f_{s,\xi,\omega,t} \le f_{s,\xi,\omega}$ for every $|t| \le 1$. From the Dominated Convergence Theorem, we conclude

$$\lim_{t\downarrow 0} \int \big(h(s,z,\xi+t\omega)+h(s,z,\xi) \big)\, d\mu(z) \;=\; 2\int h(s,z,\xi)\, d\mu(z)\,. \tag{A.18}$$

Next, consider

$$\left| \frac{h(s,z,\xi+t\omega)-h(s,z,\xi)}{t} \right| \;\le\; \left| e^{-s\sum_j (z_j-\xi_j)^2} \right| \left| \frac{e^{-st\left(t\sum_j(\omega_j)^2 + 2\sum_j (z_j-\xi_j)\,\omega_j\right)} - 1}{t} \right|.$$

For any given complex number α we have

$$\left| \frac{e^{-st\alpha}-1}{t} \right| \;\le\; \left| \frac{e^{-st\mathrm{Re}\,\alpha}-1}{t} \right| \;\le\; e^{s|\mathrm{Re}\,\alpha|}\,.$$

Using this elemetary remark, simple calculations lead to

$$\left| \frac{h(s,z,\xi+t\omega)-h(s,z,\xi)}{t} \right| \;\le\; Ce^{Cs}e^{-s|z-\mathrm{Re}\,\xi|^2 + Cs|z-\mathrm{Re}\,\xi|} \;\le\; Ce^{C_1 s}e^{-s|z-\mathrm{Re}\,\xi|^2/2}\,, \tag{A.19}$$

where the constants C and C_1 depend only on ξ and ω. Thus, again by the Dominated Convergence Theorem,

$$\lim_{t\downarrow 0} \int \frac{h(s,z,\xi+t\omega)-h(s,z,\xi)}{t}\, d\mu(z) \;=\; \int \frac{\partial h}{\partial \omega}(s,z,\xi)\, d\mu(z)\,. \tag{A.20}$$

Next, from (A.15), it follows easily that

$$e^{-s^2} \int \big| h(s,z,\xi+t\omega)+h(s,z,\xi) \big|\, d\mu(z) \;\le\; Ce^{Cs-s^2}\,. \tag{A.21}$$

Similarly, by (A.19), the same computations leading to (A.15) give

$$\int \left| \frac{h(s,z,\xi+t\omega)-h(s,z,\xi)}{t} \right| d\mu(z) \;\le\; Ce^{Cs}\,. \tag{A.22}$$

Hence, by (A.21) and (A.22) we conclude

$$\begin{aligned} & e^{-s^2}\int |h(s,z,\xi+t\omega)+h(s,z,\xi)|\, d\mu(z) \\ & \qquad \cdot \int \left| \frac{h(s,z,\xi+t\omega)-h(s,z,\xi)}{t} \right| d\mu(z) \;\le\; C_1 e^{Cs-s^2}\,. \end{aligned} \tag{A.23}$$

Therefore, by the Dominated Convergence Theorem, (A.18), (A.20), and (A.23) give (A.17).

$\square$

APPENDIX B

Gaussian integrals

PROPOSITION B.1.

$$\int_{\mathbb{R}^m} e^{-|z|^2}\, d\mathscr{L}^m(z) \;=\; \pi^{m/2} \tag{B.1}$$

$$\omega_{2m} \;:=\; \mathscr{L}^{2m}(B_1(0)) \;=\; \frac{\pi^m}{m!} \tag{B.2}$$

$$\omega_{2m+1} \;:=\; \mathscr{L}^{2m+1}(B_1(0)) \;=\; \frac{2^{m+1}\pi^m}{(2m+1)(2m-1)\ldots 3\cdot 1} \tag{B.3}$$

$$\int_{\mathbb{R}^m} |z|^{2j}\, e^{-|z|^2}\, d\mathscr{L}^m(z) \;=\; \left(j-1+\frac{m}{2}\right)\ldots\left(1+\frac{m}{2}\right)\frac{m}{2}\pi^{m/2} \tag{B.4}$$

$$\int_{\mathbb{R}^m} |z| e^{-|z|^2}\, d\mathscr{L}^m(z) \;=\; \frac{m\omega_m}{(m+1)\omega_{m+1}}\pi^{(m+1)/2} \tag{B.5}$$

$$\int_{\mathbb{R}^m} |z|^{2j+1}\, e^{-|z|^2}\, d\mathscr{L}^m(z) \;=\; \left(j+\frac{m-1}{2}\right)\ldots\left(1+\frac{m-1}{2}\right)\frac{m\omega_m}{(m+1)\omega_{m+1}}\pi^{(m+1)/2}\,. \tag{B.6}$$

PROOF. **(i)** Note that from Fubini's Theorem we have

$$\int_{\mathbb{R}^m} e^{-|z|^2}\, d\mathscr{L}^m(z) \;=\; \int_{\mathbb{R}^m} e^{-z_1^2-z_2^2-\ldots-z_m^2}\, dz_1\, dz_2\, \ldots dz_m \;=\; \left[\int_{\mathbb{R}} e^{-x^2}\, d\mathscr{L}^1(x)\right]^m\,. \tag{B.7}$$

When $m=2$ we obtain

$$\begin{aligned}\left[\int_{\mathbb{R}} e^{-x^2}\, d\mathscr{L}^1(x)\right]^2 \;&=\; \int_{\mathbb{R}^2} e^{-|z|^2}\, d\mathscr{L}^2 \\ &=\; \int_0^\infty 2\pi r e^{-r^2}\, dr \;=\; -\pi e^{-r^2}\Big|_0^\infty \;=\; \pi\,,\end{aligned}$$

and hence

$$\int_{\mathbb{R}} e^{-x^2}\, d\mathscr{L}^1(x) \;=\; \pi^{1/2}\,.$$

Using again (B.7) we conclude

$$\int_{\mathbb{R}^m} e^{-|z|^2}\, d\mathscr{L}^m(z) \;=\; \pi^{m/2}\,.$$

(ii) Recalling that $\mathscr{H}^{2m-1}(\partial B_r(0)) = 2mw_{2m}r^{2m-1}$, we obtain

$$\int_{\mathbb{R}^{2m}} e^{-|z|^2}\, d\mathscr{L}^{2m}(z) \;=\; \int_0^\infty 2m\omega_{2m}\, r^{2m-1}\, e^{-r^2}\, dr\,.$$

Writing $r^{2m-1}e^{-r^2} = r^{2m-2}\big(re^{-r^2}\big)$ and integrating by parts we obtain

$$\begin{aligned} 2m\omega_{2m}\int_0^\infty r^{2m-1}e^{-r^2}\,dr &= 2m\omega_{2m}(2m-2)\int_0^\infty \frac{r^{2m-3}}{2}e^{-r^2}\,dr \\ &= 2m(m-1)\omega_{2m}\int_0^\infty r^{2m-3}e^{-r^2}\,dr\,. \end{aligned}$$

By induction we have

$$\int_{\mathbb{R}^{2m}} e^{-|z|^2}\,d\mathscr{L}^{2m}(z) = m!\omega_{2m}\int_0^\infty 2re^{-r^2}\,dr = m!\omega_{2m}$$

and hence from (B.1) we conclude $\omega_{2m} = \pi^m/m!$.

(iii) Again using polar coordinates and integrating by parts we obtain

$$\int_{\mathbb{R}^{2m+1}} e^{-|z|^2}\,d\mathscr{L}^{2m+1}(z) = \omega_{2m+1}\frac{(2m+1)(2m-1)\ldots 3\cdot 1}{2^m}\int_0^\infty e^{-r^2}\,dr\,.$$

Therefore, from (B.1) we have

$$\pi^{m+1/2} = \omega_{2m+1}\frac{(2m+1)(2m-1)\ldots 3\cdot 1}{2^{m+1}}\pi^{1/2}\,,$$

from which we conclude (B.3).

(iv) It is easy to check that

$$|z|^{2j}e^{-|z|^2} = |z|^{2j-2}\langle z, ze^{-|z|^2}\rangle = -|z|^{2j-2}\left\langle z/2, \nabla_z\left(e^{-|z|^2}\right)\right\rangle\,.$$

Using this observation and integrating by parts we obtain

$$\begin{aligned} \int_{\mathbb{R}^m} |z|^{2j}e^{-|z|^2}\,d\mathscr{L}^m(z) &= \frac{1}{2}\int_{\mathbb{R}^m}\left[\operatorname{div}\left(|z|^{2j-2}z\right)\right]e^{-|z|^2}\,d\mathscr{L}^m(z) \\ &= \frac{1}{2}\int_{\mathbb{R}^m}(2j-2+m)|z|^{2j-2}e^{-|z|^2}\,d\mathscr{L}^m(z)\,. \end{aligned}$$

By induction we find

$$\int_{\mathbb{R}^m} |z|^{2j}e^{-|z|^2}\,d\mathscr{L}^m(z) = \left(j-1+\frac{m}{2}\right)\left(j-2+\frac{m}{2}\right)\ldots\left(1+\frac{m}{2}\right)\frac{m}{2}\int_{\mathbb{R}^m} e^{-|z|^2}\,d\mathscr{L}^m(z)\,.$$

Using (B.1) we conclude (B.4).

(v) Integrating by parts as above, we compute

$$\int_{\mathbb{R}^m} |z|^{2j+1}e^{-|z|^2}\,d\mathscr{L}^m(z) = \left(j+\frac{m-1}{2}\right)\ldots\left(1+\frac{m-1}{2}\right)\int |z|e^{-|z|^2}\,d\mathscr{L}^m(z)\,.$$

Using polar coordinates we conclude

$$\int_{\mathbb{R}^m} |z|e^{-|z|^2}\,d\mathscr{L}^m(z) = m\omega_m\int_0^\infty r^m e^{-r^2}\,dr\,.$$

Note that

$$\int_0^\infty r^m e^{-r^2}\,dr \;=\; \frac{1}{(m+1)\omega_{m+1}}\int_0^\infty (m+1)\omega_{m+1} r^m e^{-r^2}\,dr$$
$$= \frac{1}{(m+1)\omega_{m+1}}\int_{\mathbb{R}^{m+1}} e^{-|z|^2}\,d\mathscr{L}^{m+1}(z)\,,$$

from which we obtain

$$\int_{\mathbb{R}^m} |z|e^{-|z|^2}\,d\mathscr{L}^m(z) \;=\; \frac{m\omega_m}{(m+1)\omega_{m+1}}\pi^{(m+1)/2}\,.$$

Therefore, we conclude

$$\int_{\mathbb{R}^m} |z|^{2j+1}e^{-|z|^2}\,d\mathscr{L}^m(z) \;=\; \left(j+\frac{m-1}{2}\right)\cdots\left(1+\frac{m-1}{2}\right)\frac{m\omega_m}{(m+1)\omega_{m+1}}\pi^{(m+1)/2}\,.$$

$\square$

Bibliography

[1] Ambrosio, L.; Fusco, N.; Pallara, D. *Functions of bounded variations and free discontinuity problems*, Oxford Mathematical Monographs. Clarendon Press, Oxford, 2000.

[2] Besicovitch, A. S. *On the fundamental geometrical properties of linearly measurable plane sets of points II*, Math. Ann. **115** (1938), 296–329.

[3] Chlebík, M. *Geometric measure theory*, Thesis, Prague 1984.

[4] Chlebík, M. *Geometric measure theory: selected concepts, results and problems.* Handbook of measure theory, Vol. I, II, 1011–1036, North-Holland, Amsterdam, 2002.

[5] Dickinson, D. R. *Study or extreme cases with respect to densities of irregular linearly measurable plane sets of points*. Math. Ann. **116** (1939) 358–373.

[6] Farag, Hany M. *Unrectifiable 1-sets with moderate essential flatness satisfy Besicovitch's $\frac{1}{2}$-conjecture*. Adv. Math. **149** (2000), 89–129.

[7] Farag, Hany M. *On the $\frac{1}{2}$-problem of Besicovitch: quasi-arcs do not contain sharp saw-teeth.* Rev. Mat. Iberoamericana **18** (2002), 17–40.

[8] Federer, H. *Geometric measure theory*, volume 153 of Die Grundlehren der mathematischen Wissenschaften. Springer-Verlag New York Inc., New York, 1969.

[9] Huovinen, P. *A nicely behaved singular integral on a purely unrectifiable set.* Proc. Amer. Math. Soc. **129** (2001), 3345–3351

[10] Kirchheim, B.; Preiss, D. *Uniformly distributed measures in Euclidean spaces.* Math. Scan. **90** (2002) 152–160.

[11] Kowalski, O.; Preiss D. *Besicovitch–type properties of measures and submanifolds* J. Reine Angew. Math. **379** (1987), 115–151.

[12] Lorent, A. *Rectifiability of measures with locally uniform cube density.* Proc. London Math. Soc. (3) **86** (2003), no. 1, 153–249.

[13] Lorent, A. *A Marstrand type theorem for measures with cube density in general dimension.* Math. Proc. Cambridge Philos. Soc. **137** (2004), no. 3, 657–696.

[14] Lorent, A. *A Marstrand Theorem for measures with polytope density* Preprint

[15] Marstrand, J. M. *Some fundamental geometrical properties of plane sets of fractional dimensions.* Proc. London Math. Soc. (3) **4** (1954), 257–302.

[16] Marstrand, J. M. *Hausdorff two–dimensional measure in* 3 *space* Proc. London Math. Soc. (3) **11** (1961), 91–108.

[17] Marstrand, J. M. *The (ϕ, s) regular subset of n space*. Trans. Amer. Math. Soc. **113** (1964), 369–392.

[18] Mattila, P. *Hausdorff m regular and rectifiable sets in n–space* Trans. Amer. Math. Soc. **205** (1975), 263–274.

[19] Mattila, P. *Densities of measures on the real line.* Ann. Acad. Sci. Fenn. Ser. A I Math. **4** (1979), 53–61.

[20] Mattila, P. *Cauchy singular integrals and rectifiability in measures of the plane.* Adv. Math. **115** (1995), 1–34.

[21] Mattila, P. *Geometry of sets and measures in Euclidean spaces*, volume 44 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1995.

[22] Mattila, P.; Preiss, D. *Rectifiable measures in $\mathbb{R}^n$ and existence of principal values for singular integrals.* J. London Math. Soc. (2) **52** (1995), 482–496.

[23] Moore, E. F. *Density ratios and $(\phi, 1)$ rectifiability in n–space*, Trans. Amer. Math. Soc. **69** (1950), 324–334.

[24] Morse, A. P.; Randolph, J. F. *The Φ rectifiable subsets of the plane*, Trans. Amer. Math. Soc. **55** (1944), 236–305.

[25] Preiss, D. *Geometry of measures in $\mathbb{R}^n$: distribution, rectifiability, and densities.* Ann. of Math. **125** (1987), 537–643.

[26] Preiss, D.; Tišer, J. *On Besicovitch's $\frac{1}{2}$-problem.* J. London Math. Soc. (2) **45** (1992), 279–287.

[27] Rudin, W. *Principles of mathematical analysis.* Third edition McGraw-Hill Book Co., New York, 1976.