Math 426/576

Suggested problems

**Problem 1:** Bartle 8N

**Problem 2:** Let \((X, \mathcal{M}, \mu)\) be a measure space. For each \(n \in \mathbb{N}\), let \(f_n : X \to [0, \infty]\) be measurable. Assume that \(f_n\) decreases pointwise to \(f\) and that \(\int f_1 \, d\mu < \infty\). Prove that

\[
\lim_{n \to \infty} \int f_n \, d\mu = \int f \, d\mu.
\]

**Problem 3:** Assume Fatou’s lemma and deduce the monotone convergence theorem from it.

**Problem 4:** Let \(f(x) = x^{-1/2}\) if \(x \in (0, 1)\) and \(f(x) = 0\) otherwise. Let \(\{r_n\}_{n \geq 1}\) be an enumeration of the rationals. Set

\[
g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n).
\]

Prove that:

1. \(g \in L^1(m)\), and in particular \(g < \infty\) \(m\text{-a.e.}\) (\(m\) denotes the Lebesgue measure in \(\mathbb{R}\)).
2. \(g\) is discontinuous at every point and unbounded on every interval.
3. \(g^2 < \infty\) \(m\text{-a.e.}\) but \(g^2\) is not integrable on any interval.

**Problem 5:** Compute the following limits and justify the calculations:

1. \(\lim_{n \to \infty} \int_0^{\infty} (1 + (x/n))^{-n} \sin(x/n) \, dx\).
2. \(\lim_{n \to \infty} \int_0^1 (1 + nx^2)(1 + x^2)^{-n} \, dx\).
Problem 6: Let \( f \geq 0 \) be a Lebesgue integrable function on \([0, 1]\). If for every \( n = 1, 2, \cdots \)
\[
\int_0^1 [f(x)]^n \, dx = \int_0^1 f(x) \, dx
\]
show that there exists a Lebesgue measurable set \( A \subset [0, 1] \) such that \( f = \chi_A \) \( m \)-a.e..

Problem 7: If \( \mu(E_n) < \infty \) for \( n \in \mathbb{N} \) and \( \chi_{E_n} \to f \) in \( L^1(\mu) \), then \( f \) is ( \( \mu \)-a.e equal to) the characteristic function of a measurable set.

Problem 8: In Egoroff’s theorem the hypothesis \( \mu(X) < \infty \) can be replaced by \( |f_n| \leq g \) for all \( n \in \mathbb{N} \), where \( g \in L^1(\mu) \).

Problem 9: Let \((X, \mathcal{M}, \mu)\) be a measure space. Suppose \( |f_n| \leq g, \, g \in L^1(\mu) \) and \( f_n \to f \) in measure. Prove the
1. \( \int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu \)
2. \( f_n \to f \) in \( L^1(\mu) \).

Problem 10: Let \((X, \mathcal{M}, \mu)\) be a measure space. Suppose \( \{\nu_j\}_j \) is a sequence of positive measures. If \( \nu_j \perp \mu \) for all \( j \), then \( \sum_1^\infty \nu_j \perp \mu \); and if \( \nu_j \ll \mu \) for all \( j \), then \( \sum_1^\infty \nu_j \ll \mu \).