READING: Bartle, Chapters 4 and 5.


Problem 1 In this problem, \( m^* \) is the outer measure on \( \mathbb{R} \) defined by (12.1) from the Text (the outer measure used to define the Lebesgue \( \sigma \)-algebra); \( m \) is the Lebesgue measure on the real line on the \( \sigma \)-algebra of Lebesgue measurable sets on the line.

(i) For any real numbers \( a < b < c \) and any sets \( A_1, A_2 \) (not necessarily measurable!), if \( A_1 \subset [a,b) \) and \( A_2 \subset [b,c) \), then \( m^*(A_1 \cup A_2) = m^*(A_1) + m^*(A_2) \).

Hint: use Theorem 12.4; it cannot be applied directly, but you can first remove a \( \delta \)-neighborhood of the point \( b \), apply 12.4, and then let \( \delta \to 0 \).

(ii) For \( x, y \in [0,1) \) denote

\[
x + y := \begin{cases} 
  x + y, & \text{if } x + y < 1; \\
  x + y - 1, & \text{if } x + y \geq 1.
\end{cases}
\]

Further, let \( E + y := \{x + y : x \in E\} \) for \( E \subset [0,1) \).

Show that if \( E \subset [0,1) \) is Lebesgue measurable and \( y \in [0,1) \), then \( E + y \) is Lebesgue measurable and \( m(E + y) = m(E) \).

Hint: use part (i) and the fact that \( m^* \) and \( m \) are translation invariant.

Remark. The statement of part (ii) was used in the construction of the Vitali non-measurable set (see Chapter 17).