Problem 1. Suppose that \( \mu \) is a Radon measure in \( \mathbb{R}^n \). If \( \varphi \in L^1(\mu) \) and \( \varphi \geq 0 \), show that the measure \( \nu \) defined by
\[
\nu(E) = \int_E \varphi \, d\mu
\]
is also Radon.

Problem 2. Suppose that \( \mu \) is a finite Radon measure in \( \mathbb{R}^n \) such that \( \mu(\{x\}) = 0 \) for each \( x \in \mathbb{R}^n \). For any \( \alpha \) such that \( 0 < \alpha < \mu(\mathbb{R}^n) \) there exists a Borel set \( B \) such that \( \mu(B) = \alpha \).

Problem 3. Let \( (X, \mu) \) be a measure space and let \( 1 \leq p < \infty \). Suppose that \( \{f_n\} \) is a bounded sequence in \( L^p(\mu) \), and that \( f_n \rightarrow f \) \( \mu \)-a.e. on \( X \).

1. Show that \( f \in L^p(\mu) \).
2. Give an example where such sequence \( \{f_n\} \) does not converge to \( f \) in \( L^p(\mu) \).
3. Show that if \( |f_n| \leq |f| \) on \( X \) for every \( n \), then \( f_n \rightarrow f \) in \( L^p(\mu) \).

Problem 4. Suppose that \( f \) is absolutely continuous on \((0,1)\), and that \( f' \in L^p \), where \( 1 < p \leq \infty \). Let \( p^{-1} + q^{-1} = 1 \).

1. Suppose \( 1 < p < \infty \). Show that for all \( a \in (0,1) \)
\[
(1) \quad \lim_{x \to a} \frac{f(x) - f(a)}{|x - a|^{1/q}} = 0.
\]
2. Suppose that \( p = \infty \), so \( q = 1 \). Show that in this case (1) is false in general. Identify those \( f \) for which it is true.

Problem 5. Show that an orthonormal set \( \{\phi_n\}_{n \geq 1} \) in \( L^2(a,b) \) is a basis if and only if for all \( x \in (a,b) \)
\[
\sum_{n=1}^{\infty} \left| \int_a^x \phi_n(t) \, dt \right|^2 = x - a.
\]