Covering Theorems

We present Vitali’s Covering Theorem, which plays a crucial role in the study of differentiation properties of functions and measures in Euclidean spaces.

Notation: If $B$ is a closed ball in $\mathbb{R}^n$ of center $x$ and radius $r$ then $\hat{B}$ denotes the closed ball of center $x$ and radius $5r$.

Definitions:

1. A collection $F$ of closed balls in $\mathbb{R}^n$ is a cover of a set $A \subset \mathbb{R}^n$ if
   \[ A \subset \bigcup_{B \in F} B \]

2. $F$ is a fine cover of $A$ if, in addition,
   \[ \inf \{ \text{diam} B | x \in B, B \in F \} = 0 \]
   for each $x \in A$.

Vitali’s Covering Theorem:

Let $F$ be any collection of non-degenerate closed balls in $\mathbb{R}^n$ with
   \[ \sup \{ \text{diam} B | B \in F \} < \infty. \]
Then there is a countable family $G$ of disjoint balls in $F$ such that
   \[ \bigcup_{B \in F} B \subset \bigcup_{B \in G} \hat{B} \]
Corollary 1: Assume that $\mathcal{F}$ is a fine cover of $A$ by closed balls and
\[
\sup \{ \text{diam } B | B \in \mathcal{F} \} < \infty.
\]
Then there is a countable family $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that for each finite subset $\{B_1, B_2, \ldots, B_m\} \subset \mathcal{F}$, we have
\[
A \setminus \bigcup_{k=1}^{m} B_k \subset \bigcup_{B \in \mathcal{G} \setminus \{B_1, B_2, \ldots, B_m\}} B.
\]

Corollary 2: Let $U \subset \mathbb{R}^n$ be an open set. Let $\delta > 0$. There exists a countable collection $\mathcal{G}$ of disjoint closed balls in $U$ such that $\text{diam } B \leq \delta$ for all $B \in \mathcal{G}$ and
\[
\mathcal{L}^n \left( U \setminus \bigcup_{B \in \mathcal{G}} B \right) = 0.
\]

Corollary 3: Let $E \subset \mathbb{R}^n$ be a measurable set such that $\mathcal{L}^n(E) < \infty$. Let $\mathcal{F}$ be a fine cover of $E$. Given $\epsilon > 0$ there is a finite disjoint subcollection $\{B_1, B_2, \ldots, B_m\}$ of $\mathcal{F}$ such that
\[
\mathcal{L}^n \left( E \setminus \bigcup_{j=1}^{m} B_j \right) < \epsilon.
\]