Math 525

Homework due 01/31/07

Reading:
Sections 1, 2, and 3 Chapter 8 from Royden. Sections 1 and 2, Chapter 4 from Folland.

Problems from Folland:
Chapter 3, Section 4: problems 25 and 26.

Problem 1. Show that there exists a constant $C > 0$ such that for every $f \in L^1_{\text{loc}}(\mathbb{R}^n)$

$$m \left( \{ x \in \mathbb{R}^n : Mf(x) > \alpha \} \right) \leq \frac{C}{\alpha} \int_{\{ x : |f(x)| \geq \frac{\alpha}{2} \}} |f(y)| \, dy.$$ 

Recall that for $x \in \mathbb{R}^n$

$$Mf(x) = \sup_{r > 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y)| \, d(y).$$

Problem 2. Let $p \geq 1$, and let $f : \mathbb{R}^n \to [0, \infty)$ be measurable. Let

$$A_t = \{ x \in \mathbb{R}^n : f(x) > t \}.$$ 

Show that

$$\int_{A_0 \times (0, \infty)} t^{p-1} \chi_{A_t} \, dx \, dt = \int_0^\infty t^{p-1} m(A_t) \, dt = \frac{1}{p} \int_{A_0} f^p(x) \, dx.$$
Problem 3. For $p > 1$, define
\[ L^p(R^n) = \left\{ f : R^n \rightarrow R : \int_{R^n} |f(y)|^p \, dy < \infty \right\}. \]

Show that there exists a constant $C(p) > 1$ depending on $p$ such that
\[ \left( \int_{R^n} [Mf(x)]^p \, dx \right)^{\frac{1}{p}} \leq C(p) \left( \int_{R^n} |f(x)|^p \, dx \right)^{\frac{1}{p}}, \]
for all $f \in L^p(R^n)$. (Hint: Use problems 1 and 2.)

Problem 4. Let $L^n$ denote Lebesgue measure on $R^n$ and let $\mu$ be a Borel regular measure on $R^n$. Assume that for all $x \in R^n$
\[ \limsup_{r \to 0} \frac{\mu(B(x, r))}{\omega_n r^n} \geq 1, \]
where $\omega_n$ denotes the Lebesgue measure of the ball of center 0 and radius 1 in $R^n$. Prove that for any Borel subset $A \subset R^n$
\[ \mu(A) \geq L^n(A). \]

*Problem. A positive Radon measure $\mu$ on $R^n$ is said to be a doubling measure if there exist $C_0 > 1$ such that
\[ \mu(B(x, 2r)) \leq C_0 \mu(B(x, r)), \quad \forall x \in R^n \quad \text{and} \quad \forall r > 0. \]

For $f \in L^1_{loc}(\mu)$ define the Hardy-Littlewood maximal function $Mf$ by
\[ Mf(x) = \sup_{r > 0} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(y)| \, d\mu(y). \]

Let $\mu$ be a doubling measure on $R^n$. Show that there exists a constant $C > 0$ such that for all $f \in L^1(\mu)$ and all $\alpha > 0$,
\[ \mu \left( \{ x \in R^n : Mf(x) > \alpha \} \right) \leq \frac{C}{\alpha} \int_{R^n} |f(y)| \, d\mu(y). \]