Reading: Chapter 5 of Royden. Sections 5 and 6, Chapter 11 form Royden. Sections 1, 2, 4 and 5, Chapter 3 from Folland.

Problems from Folland:

Chapter 3, Section 1: problems 2, 3.

Chapter 3, Section 2: problems 8, 11.

Problem 1. For $f \in L^1_{loc}(\mathbb{R}^n)$, and $0 < s < n$ define

$$\Lambda_s = \{x \in \mathbb{R}^n : \liminf_{r \to 0} \frac{1}{r^s} \int_{B(x,r)} |f(y)| dy > 0\}.$$ 

Prove that $m(\Lambda_s) = 0$, for all $s \in (0, n)$.

*Problem. A positive measure $\mu$ on $\mathbb{R}^n$ is said to be a doubling measure if there exist $C_0 > 1$ such that

$$\mu(B(x, 2r)) \leq C_0 \mu(B(x, r)), \quad \forall x \in \mathbb{R}^n \text{ and } \forall r > 0.$$ 

For $f \in L^1_{loc}(\mu)$ define the Hardy-Littlewood maximal function $Mf$ by

$$Mf(x) = \sup_{r > 0} \frac{1}{\mu(B(x, r))} \int_{B(x,r)} |f(y)| d\mu(y).$$

Let $\mu$ be a doubling measure on $\mathbb{R}^n$. Show that there exists a constant $C > 0$ such that for all $f \in L^1(\mu)$ and all $\alpha > 0$,

$$\mu(\{x \in \mathbb{R}^n : Mf(x) > \alpha\}) \leq \frac{C}{\alpha} \int_{\mathbb{R}^n} |f(y)| d\mu(y).$$