Math 524

Homework due 11/06/02

**Reading:** Section 3 Chapter 1 in Folland. Section 1 Chapter 2 in Folland.

**Problem 1.**  **(Prelim)** Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of continuously differentiable functions (i.e. $f_n \in C^1([0, 1])$) which satisfy $f_n(0) = 0$, and

$$
\int_0^1 |f_n'(x)|^2 \, dx \leq 1 \quad \text{for all } n \in \mathbb{N}.
$$

Prove that there is a subsequence of $f_n$ which converges uniformly on $[0, 1]$.

Hint: recall that if $a, b \geq 0$ and $\epsilon > 0$ then $2ab \leq a^2/\epsilon + \epsilon b^2$.

**Problem 2.** Let $X = \{ f : [0, 1] \to \mathbb{R}; f \text{ piecewise continuous} \}$. We say that $f$ is piecewise continuous on $[0, 1]$ if $f$ has only finitely many discontinuities in $[0, 1]$. For $f, g \in X$ we say that $f \sim g$ is $f - g \equiv 0$ on $[0, 1]$, except for finitely many points. This defines an equivalence relation on $X$. Let $Y = \{ [f] : f \in X \}$ where $[f]$ denotes the equivalence class of $f \in X$. For $[f], [g] \in Y$ define

$$
d([f], [g]) = \int_0^1 |f - g| \, dx.
$$

Show that $d$ defines a metric on $Y$. Show that the closed ball of radius 1 and center $[0]$, i.e.

$$
\{ [f] \in Y : d([f], [0]) \leq 1 \}
$$

is not compact.

**Problems from Folland:**

Chapter 1, Section 3: problems 9, 10.
Chapter 2, Section 1: problems 1, 2, 9.
**Problem:** Caratheodory’s criterion
Let \( \mu \) be a measure on \( \mathbb{R}^n \). If

\[
\mu(A \cup B) = \mu(A) + \mu(B), \quad \forall \ A, B \subset \mathbb{R}^n \text{ with } d(A, B) > 0,
\]

then \( \mu \) is a Borel measure (i.e. Borel sets are \( \mu \)-measurable). Here

\[
d(A, B) = \inf\{ |a - b| : a \in A, b \in B \}.
\]